Surreal Numbers – An Introduction

Version 1.6

Claus Tøndering

18 September 2013

Copyright and disclaimer

This document is Copyright ©2013 by Claus Tøndering.
E-mail: claus@tondering.dk. (Please include the word "surreal" in the subject line.)
The document may be freely distributed, provided this copyright notice is included and no money is charged for the document.

This document is provided "as is". No warranties are made as to its correctness.

The latest version of this document is available on the internet at
`http://www.tondering.dk/claus/surreal.html`

Changes since version 1.5

Several typographical errors corrected.

The wording of theorem 9 on page 20 has been changed.

A clarification was added to the definition of $\frac{1}{x}$ on page 43.

The definition of surreal exponentiation on page 44 has been changed.

Contents

Preface

My interest in surreal numbers was first awakened when I came across the subject in a lecture by Donald Knuth. I started looking around the internet for information about surreal numbers, but what I found was no more than brief overviews and sketchy introductions that left me with even more questions.

But I got enough information to start doing some research myself, and the subject grew on me and became more and more fascinating. And, of course, I eventually got round to reading some of the good books on the subject, all of which are listed in chapter 7.

The present paper serves a two-fold purpose. It is the result of my wanting to commit to paper much of what I have learned about surreal numbers, and at the same time I hope it will provide other users of the internet with a free and accessible introduction to a very fascinating subject in pure mathematics. I am not a mathematician by profession, and many of the more advanced sides to the subject are inaccessible to me. Nevertheless, I hope that the readers will find the paper useful and enjoyable.

I welcome comments. I can be reached by email at claus@tondering.dk (please include the word "surreal" in the subject line).

Finally, I hope that this paper may glorify God by showing – once again – the beauty of His world, a beauty that is demonstrated here by the wealth that springs out of pure logic.

Prerequisites

The reader of this document is expected to have a good working knowledge of set theory and formal logic expressions. The list below is a description of the symbolism used in this paper, and at the same time it can serve as a short reminder for those of you whose formal math is a little rusty.

You are also expected to be acquainted with proving things by induction and with indirect proofs. *Proof by induction* means, for example, proving something for $x = 1$, and then proving that if it works for some value x, it also works for $x + 1$. *Indirect proofs* work by assuming that what you want to prove is wrong, and seeing that that leads to an impossibility.

Set Theory

A set is an unordered collection of objects, known as the "members" or "elements" of the set.

The following table lists symbols used in connection with sets:

Notation	Meaning
$\{a, b, c\}$	A set consisting of the members a, b, and c.
$\emptyset$	The empty set.
$a \in B$	a is a member of the set B.
$A \cup B$	The union of the sets A and B. That is, a set consisting of all objects that are members of either A or B (or both).
$A \cap B$	The intersection of the sets A and B. That is, a set consisting of all objects that are members of both A and B.
$A \setminus B$	Set subtraction. That is, a set consisting of all objects that are members of A, except those that are members of B.

Formal Logic

In the following expressions, p and q are truth values (also known as "Boolean values"), that is, expressions that are either *true* or *false*.

Notation	Meaning
$p \wedge q$	Both p and q are true.
$p \vee q$	Either p or q (or both) is true.
$\neg p$	p is false.
$p \Rightarrow q$	If p is true, then q is true.
$p \Leftarrow q$	p is true if q is true.
$p \Leftrightarrow q$	p is true if and only if q is true. That is, p and q are either both true or both false.
$\forall x \in A : x > 2$	All members of the set A are greater than 2.
$\exists x \in A : x > 2$	There exists a member of the set A which is greater than 2.

The following equivalences are often useful:
$\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.
$\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
$p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$.
$\neg\forall x \in A : x > 2$ is equivalent to $\exists x \in A : \neg(x > 2)$.
$\neg\exists x \in A : x > 2$ is equivalent to $\forall x \in A : \neg(x > 2)$.
$\forall x \in A : x > 2$ is equivalent to $x \in A \Rightarrow x > 2$.

Additional symbols

The symbol $\square$ is used to indicate the end of a proof.

Chapter 1

Introduction

Surreal numbers are fascinating for several reasons. They are built on an extremely simple and small foundation, and yet they provide virtually all of the capabilities of ordinary real numbers. With surreal numbers we are able to (or rather, required to) actually *prove* things we normally take for granted, such as "$x = x$" or "$x = y \Rightarrow x + z = y + z$". Furthermore, surreal numbers extend the real numbers with a tangible concept of *infinity* and *infinitesimals* (numbers that are smaller than any positive real number, and yet are greater than zero). With surreal numbers it makes sense to talk about "infinity minus 3","infinity to the third power", or "the square root of infinity".

Surreal numbers were invented (some prefer to say "discovered") by John Horton Conway of Cambridge University and described in his book *On Numbers and Games* [1]. Conway used surreal numbers to describe various aspects of game theory, but the present paper will only briefly touch on that in chapter 6. The term "surreal number" was invented by Donald Knuth [2].

What can surreal numbers be used for? Not very much at present, except for some use in game theory. But it is still a new field, and the future may show uses that we haven't thought of. Nevertheless, surreal numbers are worth studying for two reasons. First, as a study in pure math they are a fascinating – even beautiful – subject. Secondly, they provide a good introduction to and exercise in abstract algebra, and as such they serve a didactic purpose.

So what are surreal numbers? Before we start looking at the definition, you must forget everything you know about numbers. You don't know what "less than" means. You don't know what "equal to" means. You don't know what "one" or "two" or "three" means. You don't know what addition and multiplication are. Okay?

The definition of surreal numbers is very peculiar:

Definition 1. *A surreal number is a pair of sets of previously created surreal numbers. The sets are known as the "left set" and the "right set". No member of the right set may be less than or equal to any member of the left set.*

This defines surreal numbers in terms of other surreal numbers and the concept of one number being "less than or equal to" another number. Not very promising, since we don't know what a surreal number is and we don't know what "less than or equal to" means.

We need a definition of "less than or equal to":

Definition 2. *A surreal number x is less than or equal to a surreal number y if and only if y is less than or equal to no member of x's left set, and no member of y's right set is less than or equal to x.*

There you have it: "Less than or equal to" defined in terms of "less than or equal to"! (Don't worry if this definition doesn't seem to make much sense to you. It will shortly.)

These two definitions say it all. I could stop now and omit the rest of this chapter and all of the next chapter, because everything in these two chapters is based on these two definitions, and in theory you should be able to derive all the contents of the chapters by yourself. This is the beauty of surreal numbers: Two recursive definitions that lead to an incredible wealth of structure.

But let us explore things a bit further, and let us start by taking a closer look at the definitions.

Definition 1 says that a surreal number is a pair of sets of surreal numbers. Let L and R be two sets of surreal numbers. We can then try to construct a new surreal number based on these two sets. We will write the new surreal number thus: $\{L \mid R\}$. But definition 1 imposes an additional requirement on the members of L and R: No member of R may be less than or equal to any member of L:

$$\forall\, x \in L\ \forall\, y \in R:\ y \not\leq x. \tag{1.1}$$

We will say that (1.1) specifies what is meant by a "well-formed" pair of sets. Only well-formed pairs form surreal numbers.[1]

Notational convention. *The symbol $\leq$ means "less than or equal to". The symbol $\not\leq$ means "not less than or equal to". Thus, $x \not\leq y$ is equivalent to $\neg(x \leq y)$.*

How can we use this for anything? Well, since we don't know any surreal numbers yet, we'll have to start with L and R both being the empty set. So our first surreal number is $\{\emptyset \mid \emptyset\}$.

At this point we'll make a notational shortcut: We will never actually write the symbol $\emptyset$ unless we absolutely have to. So we'll write our first surreal number thus:

$$\{\,\mid\,\}$$

Is this pair well-formed? If it isn't, it's not a surreal number. So, are any members of the right set less than or equal to any members of the left set? We still don't know what "less than or equal to" means, but since both the sets are empty, it doesn't really matter. Of course, empty sets don't contain members that can violate a requiment. So, fortunately, $\{\,\mid\,\}$ is well-formed.

We will choose a name for $\{\,\mid\,\}$; we will call it "zero" and we will denote it by the symbol 0:

$$0 \equiv \{\,\mid\,\}. \tag{1.2}$$

I use the symbol $\equiv$ to indicate that two objects are identical. 0 is simply a shorthand way to write $\{\,\mid\,\}$. Note that $\equiv$ is not the same as $=$. We haven't yet defined what it means for two surreal numbers to be equal, so we don't know what $=$ means.

Now let's turn to definition 2. We will investigate if our surreal number is less than or equal to itself. In other words, we want to prove that

$$\{\,\mid\,\} \leq \{\,\mid\,\} \tag{1.3}$$

[1] Sometimes it will be useful to say that the surreal number itself is well-formed, even though – strictly speaking – this is superfluous, because if it weren't well-formed, it wouldn't be a surreal number.

Note that with ordinary real numbers it may seem pretty silly to try to prove that $0 \le 0$, but we must not let knowledge about real numbers lead us to assume anything about surreal numbers. So now we will proceed to prove that $0 \le 0$.

Proof. Definition 2 states that

$$x \le y \quad \Leftrightarrow \quad \neg\exists\, x_L \in X_L : \, y \le x_L \quad \wedge \quad \neg\exists\, y_R \in Y_R : \, y_R \le x, \tag{1.4}$$

where X_L and Y_R are the left set of x and the right set of y, respectively.

Therefore, in order to prove (1.3) we need to prove two things:

$$\neg\exists\, x_L \in \emptyset : \, \{\,|\,\} \le x_L \tag{1.5}$$

and

$$\neg\exists\, y_R \in \emptyset : \, y_R \le \{\,|\,\}. \tag{1.6}$$

That's easy! Of course, there doesn't exist any member in the empty set that satisfies anything. So even though we still don't know what "less than or equal to" means, we can still say with certainty that (1.5) and (1.6) are true. Therefore we have now proved (1.3); or, if you like, we have proved that $0 \le 0$. □

Now that we know one surreal number, we can use this to create some new ones, because we can let 0 be a member of the left or the right set of a new surreal number.

To make things easier to read, we'll make another notational shortcut. If, for example, the left set is $\{0\}$ and the right set is $\emptyset$, we should really write our new surreal number thus: $\{\{0\}\,|\,\emptyset\}$. But as this is rather hard to read, we will not only omit the $\emptyset$, we will also discard the extra set of curly braces. So instead we'll simply write $\{0\,|\,\}$.

With either the left set or the right set equal to $\{0\}$, we can create three new numbers:

$$\{0\,|\,\},\ \{\,|\,0\},\text{ and }\{0\,|\,0\}.$$

The last of these three is not well-formed, because $0 \le 0$. So $\{0\,|\,0\}$ is not a surreal number. It is what we'll call a pseudo-number, and we'll get back to those in chapter 6. The two other numbers are well-formed, however; so we'll take a closer look at them.

First, we'll prove that

$$\{\,|\,\} \le \{0\,|\,\}. \tag{1.7}$$

Proof. According to definition 2 as expressed in (1.4), (1.7) is true if both

$$\neg\exists\, x_L \in \emptyset : \, \{0\,|\,\} \le x_L \tag{1.8}$$

and

$$\neg\exists\, y_R \in \emptyset : \, y_R \le \{\,|\,\} \tag{1.9}$$

are true.

As before, this is trivially true, since no member of the empty set satisfies any relation. □

We can also prove that

$$\{0\,|\,\} \not\leq \{\,|\,\}. \tag{1.10}$$

Proof. (1.10) is true if either

$$\exists\, x_L \in \{0\} : \{\,|\,\} \leq x_L \tag{1.11}$$

or

$$\exists\, y_R \in \emptyset : y_R \leq \{0\,|\,\} \tag{1.12}$$

is true.

(1.11) turns out to be true, because $\{\,|\,\} \leq 0$ as stated by (1.3). Therefore (1.10) is true. □

We can also prove that

$$\{0\,|\,\} \leq \{0\,|\,\}. \tag{1.13}$$

Proof. According to (1.4), (1.13) is true if both

$$\neg\exists\, x_L \in \{0\} : \{0\,|\,\} \leq x_L \tag{1.14}$$

and

$$\neg\exists\, y_R \in \emptyset : y_R \leq \{0\,|\,\} \tag{1.15}$$

are true.

(1.15) is trivially true, and (1.14) is true because $\{0\,|\,\} \not\leq 0$ according to (1.10). □

The time has come to define some new shorthand notations:

Notational convention. *Instead of $x \leq y$, we may write $y \geq x$. The symbol $\geq$ shall be read "greater than or equal to". Its opposite, "not greater than or equal to", will be written $\not\geq$.*

Instead of $x \leq y \wedge y \not\leq x$, we may write $x < y$. The symbol $<$ shall be read "less than". Its opposite, "not less than", will be written $\not<$.

Instead of $x < y$, we may write $y > x$. The symbol $>$ shall be read "greater than". Its opposite, "not greater than", will be written $\not>$.

Instead of $x \leq y \wedge y \leq x$, we may write $x = y$. The symbol $=$ shall be read "equal to". Its opposite, "not equal to", shall be written $\neq$ (and is equivalent to $x \not\leq y \vee y \not\leq x$).

You may think that this is quite trivial, but it most certainly isn't. You must not be tempted by what you know about ordinary real numbers to assume anything about surreal numbers. We want very strict definitions of these relations.

With these new notational conventions, we can write (1.3) thus:

$$0 = 0. \tag{1.16}$$

We can write (1.7) and (1.10) thus:

$$0 < \{0\,|\,\}. \tag{1.17}$$

And we can write (1.13) thus:

$$\{0\,|\,\} = \{0\,|\,\}. \tag{1.18}$$

In a similar manner it is easy to prove the following:

$$\{\,|\,0\} < 0 \tag{1.19}$$
$$\{\,|\,0\} = \{\,|\,0\} \tag{1.20}$$
$$\{\,|\,0\} < \{0\,|\,\}. \tag{1.21}$$

We will choose appropriate names for $\{0\,|\,\}$ and $\{\,|\,0\}$: We will call $\{0\,|\,\}$ "one" and we will denote it by the symbol 1, and we will call $\{\,|\,0\}$ "minus one" and denote it by the symbol -1:

$$1 \equiv \{0\,|\,\} \tag{1.22}$$
$$-1 \equiv \{\,|\,0\}. \tag{1.23}$$

The many relations we have proved about these two surreal numbers now amount to:

$$0 < 1 \tag{1.24}$$
$$1 = 1 \tag{1.25}$$
$$-1 < 0 \tag{1.26}$$
$$-1 = -1 \tag{1.27}$$
$$-1 < 1. \tag{1.28}$$

Amazing, isn't it?

We now know three surreal numbers. By using them as members of the left and right set of new numbers we can come up with no less than 61 new numbers. Most of these, such as $\{1\,|-1\}$, are not well-formed, but we are left with these 17 new well-formed numbers:

$$\begin{aligned}
&\{-1\,|\,\},\ \{\,|-1\},\\
&\{1\,|\,\},\ \{\,|\,1\},\\
&\{-1,0\,|\,\},\ \{-1\,|\,0\},\ \{\,|-1,0\},\\
&\{0,1\,|\,\},\ \{0\,|\,1\},\ \{\,|\,0,1\},\\
&\{-1,1\,|\,\},\ \{-1\,|\,1\},\ \{\,|-1,1\},\\
&\{-1,0,1\,|\,\},\ \{-1,0\,|\,1\},\ \{-1\,|\,0,1\},\ \{\,|-1,0,1\}.
\end{aligned}$$

You will notice that several of these have more than one member in either the left or the right set.

It is easy to prove that

$$1 < \{1\,|\,\} \tag{1.29}$$

and that

$$\{\,|-1\} < -1, \tag{1.30}$$

so we'll call these new numbers "two" and "minus two", respectively:

$$2 \equiv \{1\,|\,\} \tag{1.31}$$
$$-2 \equiv \{\,|\,-1\}. \tag{1.32}$$

More interestingly, we can prove that

$$0 < \{0\,|\,1\} \quad \text{and} \quad \{0\,|\,1\} < 1 \tag{1.33}$$

and that

$$-1 < \{-1\,|\,0\} \quad \text{and} \quad \{-1\,|\,0\} < 0. \tag{1.34}$$

We will therefore call these two new numbers "one half" and "minus one half", respectively:

$$\tfrac{1}{2} \equiv \{0\,|\,1\} \tag{1.35}$$
$$-\tfrac{1}{2} \equiv \{-1\,|\,0\}. \tag{1.36}$$

You may well ask why we use $\frac{1}{2}$. Why not $\frac{1}{3}$ or $\frac{14}{17}$? The answer will be found in chapter 3 when we'll discover that $\{0\,|\,1\} + \{0\,|\,1\} = 1$, so "one half" seems an appropriate name.

This takes care of four of our new surreal numbers.

Something amazing happens when we take a look at $\{-1\,|\,1\}$. We will first prove that

$$\{-1\,|\,1\} \leq \{\,|\,\}. \tag{1.37}$$

Proof. (1.37) is true if both of these statements are true:

$$\neg\exists\, x_L \in \{-1\} : \ \{\,|\,\} \leq x_L \tag{1.38}$$

and

$$\neg\exists\, y_R \in \emptyset : \ y_R \leq \{-1\,|\,1\}. \tag{1.39}$$

(1.38) is true because $0 \not\leq -1$, and (1.39) is trivially true. □

In a similar manner it is easy to prove that

$$\{\,|\,\} \leq \{-1\,|\,1\}. \tag{1.40}$$

If we take (1.37) and (1.40) together, we find that

$$\{\,|\,\} = \{-1\,|\,1\}. \tag{1.41}$$

Note that we use $=$ here, not $\equiv$. The constituent sets of $\{\,|\,\}$ and $\{-1\,|\,1\}$ are very much different, so the two numbers are not identical, which would warrant the use of $\equiv$. What (1.41) says is that $\{\,|\,\}$ and $\{-1\,|\,1\}$ have the same value, namely 0. We say that $\{\,|\,\}$ and $\{-1\,|\,1\}$ are two different *representatives* of the number zero.

Similarly, we can prove that:

$$\begin{aligned}
\{-1\,|\,\} &= 0\\
\{\,|\,1\} &= 0\\
\{-1,0\,|\,\} &= 1\\
\{\,|\,-1,0\} &= -2\\
\{0,1\,|\,\} &= 2\\
\{\,|\,0,1\} &= -1\\
\{-1,1\,|\,\} &= 2\\
\{\,|\,-1,1\} &= -2\\
\{-1,0,1\,|\,\} &= 2\\
\{-1,0\,|\,1\} &= \tfrac{1}{2}\\
\{-1\,|\,0,1\} &= -\tfrac{1}{2}\\
\{\,|\,-1,0,1\} &= -2.
\end{aligned}$$

So none of these give us any new values.

You may also notice another principle here:

$$\begin{aligned}
\{-1,0,1\,|\,\} &= 2\\
\{0,1\,|\,\} &= 2\\
\{-1,1\,|\,\} &= 2\\
\{1\,|\,\} &= 2.
\end{aligned}$$

We will prove in the next chapter that when determining the value of a surreal number we need only consider the highest value of the left set (1 in this case) and the lowest value of the right set.

We now know these surreal numbers: -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, and 2. Again, we can create new surreal numbers based on these. Proceeding as before, we will discover eight new surreal numbers that we will name thus:

$$\begin{aligned}
-3 &\equiv \{\,|\,-2\}\\
-1\tfrac{1}{2} &\equiv \{-2\,|\,-1\}\\
-\tfrac{3}{4} &\equiv \{-1\,|\,-\tfrac{1}{2}\}\\
-\tfrac{1}{4} &\equiv \{-\tfrac{1}{2}\,|\,0\}\\
\tfrac{1}{4} &\equiv \{0\,|\,\tfrac{1}{2}\}\\
\tfrac{3}{4} &\equiv \{\tfrac{1}{2}\,|\,1\}\\
1\tfrac{1}{2} &\equiv \{1\,|\,2\}\\
3 &\equiv \{2\,|\,\}.
\end{aligned}$$

We will also discover a lot of new representatives for well-known values. For example,

$$\{-2,\tfrac{1}{2}\,|\,2\} = 1 \quad \text{and} \quad \{-2\,|\,1\} = 0.$$

We define the concept of the *birthday* of a surreal number. First we created 0; we will say that that number was *born on day zero*. Then, using 0, we created -1 and 1; we will say that these two numbers were *born on day one*. Then, using -1, 0, and 1 we created -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2, which were all *born on day two*.

We will also say that 0 is *older* than 1, and that 2 is *younger* than 1.

This concludes our informal introduction to surreal numbers. In the following chapter we will attack the matter in a more rigorous fashion.

Chapter 2

Basic Properties

In the remainder of this paper we will use this notational convention:

Notational convention. *Surreal numbers will be denoted by lower case letters. Sets of surreal numbers will be denoted by upper case letters.*

Furthermore, if x is a surreal number, we will use X_L and X_R to refer to, respectively, the left and right set of x.

It will be convenient for us to use a shorthand notation for the relationship between surreal numbers and sets, or between sets.

Notational convention. *For two sets of surreal numbers, A and B, and a surreal number, c, we define the relations $<$, $\leq$, $>$, $\geq$, $\not<$, $\not\leq$, $\not>$, and $\not\geq$ thus:*

$$\begin{aligned} A \leq c &\text{ if and only if } \forall\, a \in A:\ a \leq c, \\ c \leq A &\text{ if and only if } \forall\, a \in A:\ c \leq a, \\ A \leq B &\text{ if and only if } \forall\, a \in A\, \forall\, b \in B:\ a \leq b, \end{aligned}$$

and similarly for the other relations.

Example. $\{1,3,5\} < \{6,7\}$ because both 1, 3, and 5 are less than 6 and 7.

$\{3,5,6\} \not< 1$ because neither 3, 5, and 6 is less than 1.

Note that when sets are involved, $\neg(A \leq b)$ is *not* equivalent to $A \not\leq b$. For example, neither $\{3,5\} \leq 4$ nor $\{3,5\} \not\leq 4$ is true.

It follows from the definition above that $\emptyset \leq b$ is always true, regardless of the value of b; and so is $\emptyset \not\leq b$, $\emptyset > b$, etc.

Notational convention. *For two sets of surreal numbers, A and B, we define the relation $=$ thus:*

$A = B$ *if and only if* $\forall\, a \in A\, \exists\, b \in B : a = b \ \wedge\ \forall\, b \in B\, \exists\, a \in A : a = b.$

Less formally, this means that $A = B$ if the members of the two sets are equal to one another.

Example. $\{\{\,|\,\}, \{1\,|\,\}\} = \{\{-1\,|\,1\}, \{\,|\,\}, \{-1,0,1\,|\,\}\}$ because $\{\,|\,\} = \{-1\,|\,1\}$, $\{\,|\,\} = \{\,|\,\}$, and $\{1\,|\,\} = \{-1,0,1\,|\,\}$ as we saw in chapter 1.

As we did in the previous chapter, we will also make some simplifications in writing surreal numbers: When writing the left and right sets, we will simply write sets and numbers separated by commas, and we will omit writing the empty set. Thus, if the left set of a number is $\{a, b\} \cup C \cup D$ and the right set is the empty set, we will write the number as $\{a, b, C, D \,|\, \}$ rather than the less legible $\{\{a, b\} \cup C \cup D \,|\, \emptyset\}$.

Using our new notation, we can restate the well-formedness requirement of (1.1) thus:

If x is a surreal number, then

$$X_L \ngeq X_R. \tag{2.1}$$

We can also restate the definition of "less than or equal to" thus:

$$x \le y \quad \Leftrightarrow \quad y \nleq X_L \ \wedge \ Y_R \nleq x. \tag{2.2}$$

While we're at it, it is probably a good idea to present a formal definition of what the $\equiv$ relation means. We've already noted the difference between $=$ and $\equiv$ informally: $x = y$ is defined as $x \le y \wedge y \le x$, whereas $x \equiv y$ means that x and y are identical in the sense that all the members of the left and right set of x are identical to the members of the left and right set of y.

Formally, we can define $x \equiv y$ in this way:

$$\begin{aligned} x \equiv y \quad \Leftrightarrow \quad & \forall\, x_L \in X_L : \ x_L \in Y_L \quad \wedge \\ & \forall\, x_R \in X_R : \ x_R \in Y_R \quad \wedge \\ & \forall\, y_L \in Y_L : \ y_L \in X_L \quad \wedge \\ & \forall\, y_R \in Y_R : \ y_R \in X_R. \end{aligned} \tag{2.3}$$

The members of the left and right set of a surreal number are called the *parents*[1] of the number.

Notice that definition 1 states that every surreal number is created from *previously created* surreal numbers. This means that for any surreal number, you can go back through its parents, and their parents, and their parents, and so on until you reach the primeval surreal number $\{\,|\,\}$ (or "zero"), the number from which all surreal numbers ultimately descend.

Consider, for example, the number $\frac{1}{2}$ introduced in the previous chapter. Since $\frac{1}{2} \equiv \{0 \,|\, 1\}$, the parents of $\frac{1}{2}$ are 0 and 1. And the sole parent of 1 is 0.

We will frequently use this principle to prove things by induction thus:

- We will prove that a theorem is true for $\{\,|\,\}$.
- We will prove that the theorem is true for a number x if it is true for x's parents.

Now let's start proving some characteristics of surreal numbers.

Let us recap the definition of "less than or equal to":

$$x \le y \quad \Leftrightarrow \quad \neg\exists\, x_L \in X_L : \ y \le x_L \quad \wedge \quad \neg\exists\, y_R \in Y_R : \ y_R \le x. \tag{2.4}$$

We will frequently find the opposite statement useful:

$$x \nleq y \quad \Leftrightarrow \quad \exists\, x_L \in X_L : \ y \le x_L \quad \vee \quad \exists\, y_R \in Y_R : \ y_R \le x. \tag{2.5}$$

[1]Conway [1] uses the term *option* instead of parent.

Theorem 1. *If x is a surreal number, then $x \leq x$.*

Proof. We will use induction to prove this.

First, note that $0 \leq 0$ (we proved this in the previous chapter), so the theorem is true for 0, the common ancestor of all surreal numbers.

We now assume that the theorem is true for x's parents, that is, for all members of X_L and X_R. Based on this assumption we can prove that the theorem is true for x.

(2.4) states that $x \leq x$ if and only if

$$\neg\exists\, x_L \in X_L : \; x \leq x_L \quad \wedge \quad \neg\exists\, x_R \in X_R : \; x_R \leq x \tag{2.6}$$

The left half of this statement requires us to prove that $x \nleq x_L$ for any member, x_L, of X_L. By (2.5) this statement means

$$\exists\, a \in X_L : \; x_L \leq a \quad \vee \quad \exists\, b \in X_{LR} : \; b \leq x. \tag{2.7}$$

(X_{LR} is the right set of x_L.)

We have assumed that the theorem is true for all x's parents. We therefore assume that $x_L \leq x_L$. By choosing $a \equiv x_L$, the left half of (2.7) becomes true, and therefore all of (2.7) is true.

We have now proved the left half of (2.6).

The right half of (2.6) requires us to prove that $x_R \nleq x$ for any member, x_R, of X_R. By (2.5) this statement means

$$\exists\, c \in X_{RL} : \; x \leq c \quad \vee \quad \exists\, d \in X_R : \; d \leq x_R. \tag{2.8}$$

(X_{RL} is the left set of x_R.)

We have assumed that the theorem is true for x's parents. We therefore assume that $x_R \leq x_R$. By choosing $d \equiv x_R$, the right half of (2.8) becomes true, and therefore all of (2.8) is true.

We have now proved the right half of (2.6), and since we have thus proved both halves of (2.6), the theorem is true. □

Corollary 2. *If x is a surreal number, then $x = x$.*

Proof. This follows directly from the definition of $=$. □

Note that we proved theorem 1 without using the fact that surreal numbers are well-formed. (This knowledge will come in handy later.)

One of the mistakes that people are frequently tempted to make is to assume that two surreal numbers, a and b, can be used interchangeably if $a = b$. If $a \equiv b$ the two numbers are identical, and in that case it obviously doesn't matter if you refer to a or b. But what if $a = b$? For instance, we have seen that $\{-1 \,|\, 1\} = \{\,|\,\}$, but does that mean that $\{\{-1 \,|\, 1\} \,|\,\} = \{\{\,|\,\} \,|\,\}$? When we write $\{0 \,|\,\}$, does it matter which representative for 0 we use? Fortunately, things turn out to be easy.

Theorem 3. *Let A, A', B, and B' be sets of surreal numbers, and let $a_1, a_2, a_3, \ldots$ be the members of A, let $a'_1, a'_2, a'_3, \ldots$ be the members of A', let $b_1, b_2, b_3, \ldots$ be the members of B, and let $b'_1, b'_2, b'_3, \ldots$ be the members of B'. If A and A' have the same number of members and $a_i \leq a'_i$ for all i, and B and B' have the same number of members and $b_j \leq b'_j$ for all j, then*

$$\{A \,|\, B\} \leq \{A' \,|\, B'\}. \tag{2.9}$$

Proof. In order to prove (2.9), we must – according to (2.4) – prove that

$$\neg\exists\, a \in A:\ \{A' \,|\, B'\} \leq a \quad \wedge \quad \neg\exists\, b' \in B':\ b' \leq \{A \,|\, B\}. \tag{2.10}$$

We will use an indirect proof to prove the left half of (2.10). If we assume that $\exists\, a \in A$: $\{A' \,|\, B'\} \leq a$, we must (again, according to (2.4)) have for some a in A:

$$\neg\exists\, a' \in A':\ a \leq a' \quad \wedge \quad \neg\exists\, a_R \in A_R:\ a_R \leq \{A' \,|\, B'\}. \tag{2.11}$$

The first half of (2.11) is false, because regardless of our choice of a, we know from the definition of A and A' that there always exists an a' such that our a is less than or equal to that a'. Therefore the first half of (2.11) is false, hence all of (2.11) is false, and therefore the first half of (2.10) is true.

Similarly, we can prove the right half of (2.10) by assuming that it is false. If we assume that $\exists\, b' \in B':\ b' \leq \{A \,|\, B\}$, we must have for some b' in B':

$$\neg\exists\, b'_L \in B'_L:\ \{A \,|\, B\} \leq b'_L \quad \wedge \quad \neg\exists\, b \in B:\ b \leq b'. \tag{2.12}$$

The second half of (2.12) is false, because regardless of our choice of b', we know from the definition of B and B' that there always exists a b which is less than or equal to our b'. Therefore the second half of (2.12) is false, hence all of (2.12) is false, and therefore the second half of (2.10) is true.

Thus all of (2.10) is true. □

Corollary 4. *If $A = A'$ and $B = B'$, then $\{A \,|\, B\} = \{A' \,|\, B'\}$.*

Proof. $A = A'$ means that $a_i \leq a'_i \wedge a'_i \leq a_i$ for all i, and similarly for $B = B'$. Consequently, $\{A \,|\, B\} \leq \{A' \,|\, B'\} \wedge \{A' \,|\, B'\} \leq \{A \,|\, B\}$, which implies that $\{A \,|\, B\} = \{A' \,|\, B'\}$. □

This corollary is fortunate in many ways. It relieves us – at least for now – of the tiresome burden of having to be very careful about which representative of 1 we use, when writing something like $\{1 \,|\, \}$ or $1 \in A$. However, when we turn to addition and multiplication in later chapters, we have to prove once more that equality is as good as identity.

Note that we proved theorem 3 without using the fact that surreal numbers are well-formed. (As I said previously, this knowledge will come in handy later.)

Theorem 5. *A surreal number is greater than all members of its left set and less than all members of its right set:*

$$\begin{aligned} &\forall\, a \in A:\ a < \{A \,|\, B\} \\ &\forall\, b \in B:\ \{A \,|\, B\} < b. \end{aligned} \tag{2.13}$$

Or, using set inequalities:

$$A < \{A \,|\, B\} \ \wedge\ \{A \,|\, B\} < B. \tag{2.14}$$

Proof. In order to prove the first half of (2.13), the definition of $<$ (see page 9) requires us to prove these two statements:

$$\forall\, a \in A:\ a \leq \{A \,|\, B\} \tag{2.15}$$

$$\forall\, a \in A:\ \{A \,|\, B\} \not\leq a. \tag{2.16}$$

We will prove (2.15) by induction. First we notice that it is true if $A = \emptyset$.

We then note that (2.15) is equivalent to

$$\neg\exists\, a_L \in A_L:\ \{A \,|\, B\} \leq a_L \quad \wedge \quad \neg\exists\, b \in B:\ b \leq a. \tag{2.17}$$

(Just to avoid confusion here, I'll remind the reader that a_L is a member of A_L, which is the left set of a which is a member of A. Okay?)

The right half of (2.17) is true because $\{A \,|\, B\}$ is a well-formed number, and therefore no member of B is less than or equal to a member of A.

The left half of (2.17) is equivalent to

$$\forall\, a_L \in A_L:\ \{A \,|\, B\} \not\leq a_L. \tag{2.18}$$

According to (2.5) this means that

$$\forall\, a_L \in A_L:\ (\exists\, a' \in A:\ a_L \leq a' \quad \vee \quad \exists\, a_{LR} \in A_{LR}:\ a_{LR} \leq \{A \,|\, B\}). \tag{2.19}$$

If we choose $a' \equiv a$ in the left half of the parenthesized expression in (2.19), we have:

$$\forall\, a_L \in A_L:\ a_L \leq a. \tag{2.20}$$

This statement is identical to (2.15), except that the variables have been replaced by one of their left set parents. By induction we can then conclude that (2.15) is true if it is true for a surreal number whose left set is empty, and this we know to be true.

(2.16) requires that for all a in A

$$\exists\, a' \in A:\ a \leq a' \quad \vee \quad \exists\, a_R \in A_R:\ a_R \leq \{A \,|\, B\}. \tag{2.21}$$

In the left half of this statement, we can choose $a' \equiv a$, which makes the statement true because $a \leq a$.

We have now proved the first half of (2.13). The second half can be proved in a similar manner. □

Theorem 6. *(The transitive law.)* $x \leq y \ \wedge\ y \leq z \quad \Rightarrow \quad x \leq z.$

Proof. Let us assume that the theorem is false. This means that there exists surreal numbers x, y, and z, such that

$$x \leq y \quad \wedge \quad y \leq z \quad \wedge \quad x \not\leq z \tag{2.22}$$

We will define a Boolean function[2] $p(x, y, z)$ whose value is given by (2.22):

$$p(x, y, z) \quad \Leftrightarrow \quad x \leq y \ \wedge\ y \leq z \ \wedge\ x \not\leq z \tag{2.23}$$

[2] That is, a function whose value is either *true* or *false*.

(2.22) is true if all of the following statements are true:

$$\neg\exists\, x_L \in X_L : \; y \leq x_L \tag{2.24}$$
$$\neg\exists\, y_R \in Y_R : \; y_R \leq x \tag{2.25}$$
$$\neg\exists\, y_L \in Y_L : \; z \leq y_L \tag{2.26}$$
$$\neg\exists\, z_R \in Z_R : \; z_R \leq y \tag{2.27}$$
$$\exists\, x_L \in X_L : \; z \leq x_L \quad \vee \quad \exists\, z_R \in Z_R : \; z_R \leq x. \tag{2.28}$$

Let us first assume that the left half of (2.28) is true. Note that (2.24) is equivalent to

$$\forall\, x_L \in X_L : \; y \not\leq x_L. \tag{2.29}$$

If we use the same x_L in (2.29) as the one whose existence is stipulated in the left half of (2.28), we can combine this with $y \leq z$ from (2.22) to obtain:

$$y \leq z \quad \wedge \quad z \leq x_L \quad \wedge \quad y \not\leq x_L, \tag{2.30}$$

which is equivalent to $p(y, z, x_L)$.

Let us then assume that the right half of (2.28) is true. Note that (2.27) is equivalent to

$$\forall\, z_R \in Z_R : \; z_R \not\leq y. \tag{2.31}$$

If we use the same z_R in (2.31) as the one whose existence is stipulated in the right half of (2.28), we can combine this with $x \leq y$ from (2.22) to obtain:

$$z_R \leq x \quad \wedge \quad x \leq y \quad \wedge \quad z_R \not\leq y, \tag{2.32}$$

which is equivalent to $p(z_R, x, y)$.

So, consequently we have:

$$p(x, y, z) \quad \Rightarrow \quad \exists\, x_L \in X_L : \; p(y, z, x_L) \;\; \vee \;\; \exists\, z_R \in Z_R : \; p(z_R, x, y). \tag{2.33}$$

x_L and z_R are parents of x and z, respectively. We can thus reduce $p(x, y, z)$ to a case where one of its arguments have been replace by a parent. By going back through the parents of x and z we will eventually reach $\{\,|\,\}$ where the existence stipulated in (2.33) fails because $\{\,|\,\}$ has no parents. Therefore our original assumption (2.22) must be wrong, which means that the theorem is true. □

Note that we proved theorem 6 without using the fact that surreal numbers are well-formed. (This knowledge will..., but I already said that.)

Having proved theorem 6, we will feel free to write expressions such as $x \leq y \leq z$.

We want to prove that all surreal numbers are related through the $\leq$ relation.

Theorem 7. $x \not\leq y \;\; \Rightarrow \;\; y \leq x$.

Proof. $x \not\leq y$ means that one of these statements must be true:

$$\exists\, x_L \in X_L :\ y \leq x_L \tag{2.34}$$
$$\exists\, y_R \in Y_R :\ y_R \leq x. \tag{2.35}$$

If we assume that (2.34) is true, we can find an x_L such that $y \leq x_L$. We further know from theorem 5 that $x_L \leq x$. Using the transitive law, we therefore have $y \leq x$.

If we assume that (2.35) is true, we can find a y_R such that $y_R \leq x$. We further know from theorem 5 that $y \leq y_R$. Using the transitive law, we again have $y \leq x$. □

Theorem 7 allows us to simplify the definition of "less than" given on page 9. There we stated that

$$x < y \quad\Leftrightarrow\quad x \leq y \ \wedge\ y \not\leq x. \tag{2.36}$$

But since we now know that $y \not\leq x$ implies $x \leq y$, we can simplify the definition to simply

$$x < y \quad\Leftrightarrow\quad y \not\leq x. \tag{2.37}$$

and similarly for $x > y$.

We can now prove that transitive law also holds for $<$:

Theorem 8. $x < y \ \wedge\ y < z \quad\Rightarrow\quad x < z.$

Proof. If this theorem is not true, we can find numbers x, y, and z such that

$$x < y \ \wedge\ y < z \ \wedge\ x \not< z, \tag{2.38}$$

or equivalently

$$y \not\leq x \ \wedge\ z \not\leq y \ \wedge\ z \leq x. \tag{2.39}$$

Theorem 6 is equivalent to

$$a \not\leq c \quad\Rightarrow\quad a \not\leq b \ \vee\ b \not\leq c \tag{2.40}$$

for any surreal number b.

Using (2.40) on $y \not\leq x$ in (2.39) (using $a \equiv y$, $b \equiv z$, and $c \equiv x$), we have

$$(y \not\leq z \ \vee\ z \not\leq x) \ \wedge\ z \not\leq y \ \wedge\ z \leq x. \tag{2.41}$$

But this is an impossibility. Therefore theorem 8 must be true. □

We will now prove a theorem that allows us to simplify many surreal numbers.

Theorem 9. *In a surreal number $x = \{X_L \mid X_R\}$ we can remove any member, ξ, from X_L without changing the value of x, provided that X_L contains a member that is greater than ξ. Similarly, we can remove any member, η, from X_R without changing the value of x, provided that X_R contains a member that is less than η.*

Example. $\{1,2,3 \,|\, 4,5,6\} = \{1,3 \,|\, 4,6\} = \{3 \,|\, 4\}$.

Proof. We will prove the first half of the theorem. Assume that we have $x = \{x_1, x_2, \ldots \,|\, X_R\}$ and that $x_1 < x_2$. We want to prove that $\{x_1, x_2, \ldots \,|\, X_R\} = \{x_2, \ldots \,|\, X_R\}$, which means that we must prove that $\{x_1, x_2, \ldots \,|\, X_R\} \leq \{x_2, \ldots \,|\, X_R\}$ and $\{x_2, \ldots \,|\, X_R\} \leq \{x_1, x_2, \ldots \,|\, X_R\}$, which in turn requires us to prove these four statements:

$$\neg\exists\, a \in \{x_1, x_2, \ldots\} : \{x_2, \ldots \,|\, X_R\} \leq a \tag{2.42}$$

$$\neg\exists\, b \in X_R : b \leq \{x_1, x_2, \ldots \,|\, X_R\} \tag{2.43}$$

$$\neg\exists\, a' \in \{x_2, \ldots\} : \{x_1, x_2, \ldots \,|\, X_R\} \leq a' \tag{2.44}$$

$$\neg\exists\, b' \in X_R : b' \leq \{x_2, \ldots \,|\, X_R\}. \tag{2.45}$$

(2.43), (2.44), and (2.45) follow immediately from theorem 5.
(2.42) is equivalent to

$$\forall\, a \in \{x_1, x_2, \ldots\} : \{x_2, \ldots \,|\, X_R\} \not\leq a, \tag{2.46}$$

which again means that

$$\forall\, a \in \{x_1, x_2, \ldots\} : \{x_2, \ldots \,|\, X_R\} > a. \tag{2.47}$$

Of all the possible values for a in (2.47), only $a = x_1$ could possibly be a problem (all other cases are taken care of by theorem 5). For the case of $a = x_1$ we know from theorem 5 that $\{x_2, \ldots \,|\, X_R\} > x_2$, and since $x_2 > x_1 = a$, (2.47) follows from theorem 8.

We have now proved the first half of the theorem. The second half can be proved in a similar way. □

Corollary 10. *If A has a largest member a_{max}, then $\{A \,|\, B\} = \{a_{max} \,|\, B\}$. Similarly if B has a smallest member b_{min}, then $\{A \,|\, B\} = \{A \,|\, b_{min}\}$.*

Proof. Follows trivially from theorem 9. □

Now that we have corollary 10, can we change the definition of surreal numbers so that the left and right set contain at most one number? No, because corollary 10 only works if A has a largest member and B has a smallest member. If A and B are infinite in size, they may not have a largest or smallest member. We will return to infinitely large sets in chapter 5.

Theorem 11. *If a surreal number, x, is greater than all members of a set A and less than all members of a set B, then $x = \{X_L, A \,|\, X_R, B\}$.*

This theorem can also be stated thus:

$$A < x < B \quad \Rightarrow \quad x = \{X_L, A \,|\, X_R, B\}. \tag{2.48}$$

Example. We know from chapter 1 that $-1 < 0 < 1$. We also know that $0 = \{\,|\,\}$. It therefore follows from (2.48) that $0 = \{-1 \,|\, 1\}$, which is exactly what we saw in (1.41) on page 11.

Proof. To prove that $x = \{X_L, A \mid X_R, B\}$, we need to prove that $x \leq \{X_L, A \mid X_R, B\}$ and that $\{X_L, A \mid X_R, B\} \leq x$. This means that we have to prove these four statements:

$$\neg\exists\, x_L \in X_L : \{X_L, A \mid X_R, B\} \leq x_L \tag{2.49}$$

$$\neg\exists\, \beta \in X_R \cup B : \beta \leq x \tag{2.50}$$

$$\neg\exists\, \alpha \in X_L \cup A : x \leq \alpha \tag{2.51}$$

$$\neg\exists\, x_R \in X_R : x_R \leq \{X_L, A \mid X_R, B\}. \tag{2.52}$$

(2.49) and (2.52) follow directly from theorem 5. (2.50) and (2.51) follow from theorem 5 and the definition of A and B. □

On page 12 we introduced the concept of "birthdays". The number 0 was born on day zero. The numbers -1 and 1 were born on day one. Thus after day one, we had these surreal numbers:

-1, 0, 1.

On day two, the numbers -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2 were born. Day two also saw the birth of several different surreal numbers whose value already existed. Thus after day two, we had these surreal numbers:

-2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2.

Day three added the numbers -3, $-1\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{2}$, and 3, bringing the total list of surreal numbers to

-3, -2, $-1\frac{1}{2}$, -1, $-\frac{3}{4}$, $-\frac{1}{2}$, $-\frac{1}{4}$, 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{2}$, 2.

Here we begin to notice a principle: On each day a new number is added at each end of the list, and between each of the already existing numbers.

Theorem 12. *If, after day m, the following different surreal numbers exist*

$$x_1 < x_2 < x_3 < \ldots < x_n,$$

all new numbers born on day $m+1$ can be represented by:

$$\{\,\mid x_1\}, \{x_1 \mid x_2\}, \{x_2 \mid x_3\}, \ldots, \{x_{n-1} \mid x_n\}, \{x_n \mid\,\}.$$

After day $m+1$, the order of different surreal numbers will be:

$$\{\,\mid x_1\} < x_1 < \{x_1 \mid x_2\} < x_2 < \{x_2 \mid x_3\} < x_3 < \ldots < \{x_{n-1} \mid x_n\} < x_n < \{x_n \mid\,\}. \tag{2.53}$$

Proof. We will concentrate first on the first half of the theorem, which states what surreal numbers are born on day $m+1$.

About the numbers born on day $m+1$, the theorem states that:

- $\{\,|\,x_1\}$ and $\{x_n\,|\,\}$ represent values not known on day m.
- $\{x_i\,|\,x_j\}$ represent a value not known on day m if $i+1=j$.
- $\{x_i\,|\,x_j\}$ represent a value already known on day m if $i+1\neq j$.

We know from theorem 5 that $\{\,|\,x_1\} < x_1$. Since x_1 was the smallest number born on day m, $\{\,|\,x_1\}$ must represent a new value and its position in the order given is (2.53) is obvious. Similarly, we know that $x_n < \{x_n\,|\,\}$; so $\{x_n\,|\,\}$ must also represent a value unknown on day m and its position in the order given is (2.53) is obvious. This establishes the first bullet point above.

We know from theorem 5 that $x_i < \{x_i\,|\,x_{i+1}\} < x_{i+1}$. Since at day m, we knew no number between x_i and x_{i+1}, $\{x_i\,|\,x_{i+1}\}$ must represent a new number. This proves the second bullet point.

If $i+1<j$, let x_k represent the oldest surreal number in the range $x_{i+1}\ldots x_{j-1}$. (We will later see that there is only one such oldest number, but for the present, feel free to assume that x_k is just one of several equally old numbers.) We know that the parents of x_k must either be less than x_{i+1} or greater than x_{j-1}, because before x_k was born, no number in the range $x_{i+1}\ldots x_{j-1}$ existed. So we have

$$X_{k^L} \leq x_i \;\;\wedge\;\; x_j \leq X_{k^R}, \tag{2.54}$$

where X_{k^L} and X_{k^R} are the left and right sets of x_k, respectively.

We know from theorem 5 that

$$x_i < \{x_i\,|\,x_j\} < x_j. \tag{2.55}$$

If we combine (2.54) with (2.55) we have:

$$X_{k^L} < \{x_i\,|\,x_j\} < X_{k^R}. \tag{2.56}$$

Using theorem 11, (2.56) leads to:

$$\{x_i\,|\,x_j\} = \{x_i, X_{k^L}\,|\,x_j, X_{k^R}\}. \tag{2.57}$$

We can also use theorem 11 on the fact that $x_i < x_k < x_j$ to get:

$$x_k = \{x_i, X_{k^L}\,|\,x_j, X_{k^R}\}. \tag{2.58}$$

It follows from (2.57) and (2.58) that $\{x_i\,|\,x_j\} = x_k$. This proves the third bullet point. It also makes it clear that there can be only one oldest surreal number between x_i and x_j, since the value of that oldest number must necessarily be $\{x_i\,|\,x_j\}$.

The second half of the theorem, which specifies the ordering of the numbers, follows directly from theorem 5. □

Corollary 13. *If x is the oldest surreal number between a and b, then $\{a\,|\,b\} = x$.*

Example. (1.41) on page 11 tells us that $\{-1\,|\,1\} = 0$. This also follows from the fact that 0 is the oldest surreal number between -1 and 1. Similarly, we can conclude that $\{\frac{1}{4}\,|\,2\} = 1$ and $\{\frac{1}{4}\,|\,1\} = \frac{1}{2}$.

On the previous pages we have chosen names for various surreal numbers. We have given the surreal numbers names such as '0', '1', '$-\frac{1}{2}$', etc., based primarily on whether one was greater than the other. We will now try to formulate more precisely how we convert a real number into a surreal number. Such a formalized conversion will also help us in the following chapters when we try to justify the rules for addition and multiplication.

We are going to define a function that converts real numbers into surreal numbers, and for want of a better name, I will call it the "Dali" function after the great surrealist artist, Salvador Dalí. The function is designated by the Greek letter δ.

The Dali function, $\delta(x)$, is therefore a function that maps the set of real numbers, $\mathbb{R}$, into the set of surreal numbers, $\mathbb{S}$. It is defined thus:

$$\delta(x) = \begin{cases} \{\,|\,\} & \text{if } x = 0, \\ \{\delta(x-1)\,|\,\} & \text{if } x \text{ is an integer and } x > 0, \\ \{\,|\,\delta(x+1)\} & \text{if } x \text{ is an integer and } x < 0, \\ \{\delta(\frac{j-1}{2^k})\,|\,\delta(\frac{j+1}{2^k})\} & \text{if } x \text{ can be written as an irreducible fraction } \frac{j}{2^k}, \\ & \qquad \text{where } j \text{ and } k \text{ are integers, and } k > 0. \end{cases} \tag{2.59}$$

You will notice that $\delta(0)$ is the surreal number that we have hitherto called 0. Similarly $\delta(1)$, $\delta(-1)$, and $\delta(\frac{1}{2})$ are the surreal number that we have come to know by the names 1, -1, and $\frac{1}{2}$, respectively. In the future, we will continue to say that a surreal number is equal to, say, 5, when in reality we mean that the surreal number is equal to $\delta(5)$. Also, we will continue to write a surreal number as $\{4\,|\,\}$, instead of the more precise $\{\delta(4)\,|\,\}$.

It is immediately obvious from the definition of the Dali function in connection with theorem 5 that

$$x < y \;\Leftrightarrow\; \delta(x) < \delta(y), \tag{2.60}$$

so the Dali function maps the ordering of real numbers onto the ordering of surreal numbers in an intuitive way.

But what is the value of $\delta(\frac{1}{3})$? Or $\delta(\pi)$? Or $\delta(\sqrt{2})$? The Dali function is not a proper mapping of $\mathbb{R}$ into $\mathbb{S}$ if is isn't defined for real numbers such as $\frac{1}{3}$, π, or $\sqrt{2}$. These values are not covered by (2.59). We will return to them in chapter 5.

Chapter 3

Addition and Subtraction

Before we embark on adding and subtracting surreal numbers, we shall need a shorthand notation for adding numbers to members of a set:

Notational convention. *We define the sum, $n + S$, of a number, n, and a set of numbers, S, as the set obtained by adding n to every member of S.*

In a similar manner we will write $n - S$ or $S - n$ to indicate subtraction on every member of the set; and in the next chapter we will write Sn or $S \times n$ to indicate multiplication of every member of the set.

Example. Using ordinary integer arithmetic, we have

$$\begin{aligned}
&6 + \{3, 5, 8\} = \{9, 11, 14\} \\
&6 - \{3, 5, 8\} = \{3, 1, -2\} \\
&\{3, 5, 8\} - 6 = \{-3, -1, 2\} \\
&6 \times \{3, 5, 8\} = \{18, 30, 48\}.
\end{aligned}$$

Note that any arithmetic expression on the empty set yields the empty set. (When you do something to every member of the empty set, nothing happens.):

$$\begin{aligned}
&n + \emptyset = \emptyset \\
&n - \emptyset = \emptyset \\
&n\emptyset = \emptyset.
\end{aligned}$$

Notational convention. *We define the sum of two sets, S and T, as the set obtained by adding every member of S to every member of T.*

A similar notation will be used for other operations.

Example. Using ordinary integer arithmetic, we have

$$\{10, 20\} + \{3, 5, 8\} = \{13, 15, 18, 23, 25, 28\}.$$

If the same set is mentioned more than once in an arithmetic operation, the same member is used in all instances: If $S = \{1, 2\}$ and $T = \{10, 20\}$, then $S + T + S = \{12, 14, 22, 24\}$ because we use the same member of S for both the S-terms in the sum.

Definition 3. *The sum of two surreal numbers, a and b, is defined thus:*

$$a + b = \{A_L + b, a + B_L \,|\, A_R + b, a + B_R\}. \tag{3.1}$$

Again, we see how definitions about surreal numbers always end up being recursive! Here we have addition defined in terms of addition. What saves us here is the fact that $\emptyset + n = \emptyset$.

Example. Calculate $1 + \frac{1}{2}$. To do this, we must first remember that $1 \equiv \{0\,|\,\}$ and $\frac{1}{2} \equiv \{0\,|\,1\}$. So we have

$$\begin{aligned} 1 + \tfrac{1}{2} &= \{0 + \tfrac{1}{2}, 1 + 0 \,|\, \emptyset + \tfrac{1}{2}, 1 + 1\} \\ &= \{0 + \tfrac{1}{2}, 1 + 0 \,|\, 1 + 1\}. \end{aligned} \tag{3.2}$$

This expresses $1 + \frac{1}{2}$ in terms of other sums. So let's have a go at $0 + \frac{1}{2}$. Remember that $0 \equiv \{\,|\,\}$.

$$\begin{aligned} 0 + \tfrac{1}{2} &= \{\emptyset + \tfrac{1}{2}, 0 + 0 \,|\, \emptyset + \tfrac{1}{2}, 0 + 1\} \\ &= \{0 + 0 \,|\, 0 + 1\} \end{aligned} \tag{3.3}$$

Let's find $0 + 0$:

$$\begin{aligned} 0 + 0 &= \{\emptyset + 0, 0 + \emptyset \,|\, \emptyset + 0, 0 + \emptyset\} \\ &= \{\,|\,\} \\ &= 0. \end{aligned} \tag{3.4}$$

Finally something useful! $0 + 0 = 0$, wow!

So what about $0 + 1$?

$$\begin{aligned} 0 + 1 &= \{\emptyset + 1, 0 + 0 \,|\, \emptyset + 1, 0 + \emptyset\} \\ &= \{0\,|\,\} \\ &= 1. \end{aligned} \tag{3.5}$$

Similarly, we find that $1 + 0 = 1$.

This allows us to attack (3.3):

$$\begin{aligned} 0 + \tfrac{1}{2} &= \{0 + 0 \,|\, 0 + 1\} \\ &= \{0\,|\,1\} \\ &= \tfrac{1}{2}. \end{aligned} \tag{3.6}$$

Now, let's have a go at $1 + 1$:

$$\begin{aligned} 1 + 1 &= \{0 + 1, 1 + 0 \,|\, \emptyset + 1, 1 + \emptyset\} \\ &= \{1\,|\,\} \\ &= 2. \end{aligned} \tag{3.7}$$

Finally, we can go back to (3.2):

$$\begin{aligned} 1 + \tfrac{1}{2} &= \{0 + \tfrac{1}{2}, 1 + 0 \,|\, 1 + 1\} \\ &= \{\tfrac{1}{2}, 1 \,|\, 2\} \\ &= \{1\,|\,2\} \\ &= 1\tfrac{1}{2}. \end{aligned} \tag{3.8}$$

So, $1 + \frac{1}{2} = 1\frac{1}{2}$. Astonishing, isn't it?

Example. As another example, let us calculate $\frac{1}{2}+\frac{1}{2}$:

$$\begin{aligned}\tfrac{1}{2}+\tfrac{1}{2} &= \{0+\tfrac{1}{2}, \tfrac{1}{2}+0 \,|\, 1+\tfrac{1}{2}, \tfrac{1}{2}+1\} \\ &= \{\tfrac{1}{2} \,|\, 1\tfrac{1}{2}\}.\end{aligned} \tag{3.9}$$

Here, we've used the commutative law ($a+b=b+a$) which follows easily from the definition of addition.

According to corollary 13, the value of $\{\frac{1}{2} \,|\, 1\frac{1}{2}\}$ is the value of the oldest surreal number between $\frac{1}{2}$ and $1\frac{1}{2}$, which happens to be 1.

So, $\frac{1}{2}+\frac{1}{2}=1$. This, then, is the justification of the choice of the name $\frac{1}{2}$ for the surreal number $\{0 \,|\, 1\}$, which we invented on page 11.

Before we accept definition 3, we must ask ourselves two questions:

- Is the definition legal? That is, does it produce well-formed surreal numbers?
- Does the definition make sense?

So we want to prove that the result of an addition is well-formed, but before we can do that, we need to prove the following two theorems:

Theorem 14. $x \le x' \;\wedge\; y \le y' \quad\Rightarrow\quad x+y \le x'+y'$.

Theorem 15. $x+y \ge x'+y' \;\wedge\; y \le y' \quad\Rightarrow\quad x \ge x'$.

Proof. We will prove these two theorems together by induction. For easier reference, we will define two Boolean functions p and q to match the two theorems:

$$p(x,x',y,y') \;\Leftrightarrow\; (x \le x' \;\wedge\; y \le y' \quad\Rightarrow\quad x+y \le x'+y') \tag{3.10}$$

$$q(x,x',y,y') \;\Leftrightarrow\; (x+y \ge x'+y' \;\wedge\; y \le y' \quad\Rightarrow\quad x \ge x') \tag{3.11}$$

Since we still haven't proved that addition produces well-formed surreal numbers, we can't base our proof on any theorem that relies on the well-formedness of the numbers. In other words, we can only use theorems 1, 3, and 6. (I told you it would be useful to know that these theorems didn't rely on the well-formedness of surreal numbers.)

We have

$$x+y = \{X_L+y, x+Y_L \,|\, X_R+y, x+Y_R\} \tag{3.12}$$

$$x'+y' = \{X'_L+y', x'+Y'_L \,|\, X'_R+y', x'+Y'_R\} \tag{3.13}$$

If we want to prove $x+y \le x'+y'$ in theorem 14, we need to prove

$$\neg\exists\,\alpha \in \{X_L+y, x+Y_L\} :\; x'+y' \le \alpha \;\;\wedge\;\; \neg\exists\,\beta \in \{X'_R+y', x'+Y'_R\} :\; \beta \le x+y, \tag{3.14}$$

which is true if all of the following statements are true:

$$x'+y' \not\le X_L+y \tag{3.15}$$

$$x'+y' \not\le x+Y_L \tag{3.16}$$

$$X'_R+y' \not\le x+y \tag{3.17}$$

$$x'+Y'_R \not\le x+y. \tag{3.18}$$

We will prove (3.15) by assuming that it is wrong. We know that $y \le y'$. If we combine this with the opposite of (3.15) we get

$$X_L + y \ge x' + y' \quad \wedge \quad y \le y', \tag{3.19}$$

which according to $q(X_L, x', y, y')$ (which we still haven't proved) implies that $X_L \ge x'$, which is impossible because $X_L < x \le x'$. Therefore our assumption that (3.15) is false must be wrong. Hence (3.15) is true, provided theorem 15 is true, or, rather, that $q(X_L, x', y, y')$ is true.

In a similar manner we can prove that (3.16) is true if $q(Y_L, y', x, x')$ is true. And we can prove that (3.17) is true if $q(x, X'_R, y, y')$ is true. And (3.18) is true if $q(y, Y'_R, x, x')$ is true.

We will now turn to theorem 15. Because $x + y \ge x' + y'$, we know from the definition of $\ge$ that all of the following statements are true:

$$X'_L + y' \ngeq x + y \tag{3.20}$$
$$x' + Y'_L \ngeq x + y. \tag{3.21}$$
$$x' + y' \ngeq X_R + y \tag{3.22}$$
$$x' + y' \ngeq x + Y_R. \tag{3.23}$$

If we assume that theorem 15 is wrong, we can simultaneously have $y \le y'$ and $x \ngeq x'$. The statement $x \ngeq x'$ is equivalent to

$$\exists\, x'_L \in X'_L : \; x \le x'_L \quad \vee \quad \exists\, x_R \in X_R : \; x_R \le x' \tag{3.24}$$

If we combine the first half of (3.24) with $y \le y'$ and apply $p(x, x'_L, y, y')$ (which we still haven't proved), we have $x + y \le x'_L + y'$, which contradicts (3.20).

Similarly, if we combine the second half of (3.24) with $y \le y'$ and apply $p(x_R, x', y, y')$, we have $x_R + y \le x' + y'$, which contradicts (3.22).

Therefore our assumption that theorem 15 is false must be wrong. The theorem is therefore true, provided that theorem 14 is true, or, rather, that $p(x, x'_L, y, y')$ and $p(x_R, x', y, y')$ are true.

Now we can start the induction process. We have previously seen that $p(x, y, x', y')$ is true if $q(X_L, x', y, y')$, $q(Y_L, y', x, x')$, $q(x, X'_R, y, y')$, and $q(y, Y'_R, x, x')$ are true. Now we also know that $q(x, x', y, y')$ is true if $p(x, x'_L, y, y')$ and $p(x_R, x', y, y')$ are true. By combining these statements, we get that $p(x, y, x', y')$ is true if all of the following statements are true:

$$p(X_L, x'_L, y, y') \tag{3.25}$$
$$p(X_{LR}, x', y, y') \tag{3.26}$$
$$p(Y_L, y'_L, x, x') \tag{3.27}$$
$$p(Y_{LR}, y', x, x') \tag{3.28}$$
$$p(x, X'_{RL}, y, y') \tag{3.29}$$
$$p(x_R, X'_R, y, y') \tag{3.30}$$
$$p(y, Y'_{RL}, x, x') \tag{3.31}$$
$$p(y_R, Y'_R, x, x') \tag{3.32}$$

In each of these statements one or both of the two leftmost arguments have been replaced by the parents of some of the arguments. By iteration back through the ancestors of x, x', y, and y' we therefore get that $p(x, x', y, y')$ is true if both of these statements are true:

$$p(x, \emptyset, y, y') \tag{3.33}$$
$$p(\emptyset, x', y, y'). \tag{3.34}$$

This means that both of these statements must be true:

$$x \leq \emptyset \ \wedge \ y \leq y' \quad \Rightarrow \quad x + y \leq \emptyset + y' \tag{3.35}$$
$$\emptyset \leq x' \ \wedge \ y \leq y' \quad \Rightarrow \quad \emptyset + y \leq x' + y'. \tag{3.36}$$

Since the right hand side of both (3.35) and (3.36) are obviously true (if you remember the rules for + and ≤ when applied to sets), we have proved theorem 14.

And since theorem 15 is true if theorem 14 is true, we have also proved that theorem. □

These theorems were proved without relying on the well-formedness of surreal numbers.

Corollary 16. $x = x' \ \wedge \ y = y' \quad \Rightarrow \quad x + y = x' + y'$.

This follows directly from theorem 14.

This corollary is very fortunate. It means that when we calculate $2 + 3$, the result always has the same value, regardless of what representatives of 2 and 3 we use.

Theorem 17. $x < x' \ \wedge \ y \leq y' \quad \Rightarrow \quad x + y < x' + y'$.

Proof. The definition of $<$ (see page 9) states that $x < x'$ means that $x \leq x'$ and $x \not\geq x'$. (We can't use the simpler formula in (2.37), because that relies on surreal numbers being well-formed, and we still haven't proved that addition produces well-formed numbers.)

From theorem 14 we know that

$$x + y \leq x' + y' \tag{3.37}$$

because $x \leq x'$ and $y \leq y'$.

The inverse of theorem 15 states that

$$x \not\geq x' \quad \Rightarrow \quad x + y \not\geq x' + y' \ \vee \ y \not\leq y', \tag{3.38}$$

and since we know that $y \leq y'$, we have

$$x + y \not\geq x' + y'. \tag{3.39}$$

If we combine (3.37) and (3.39) we have $x + y < x' + y'$. □

Now we can turn to proving that addition produces well-formed numbers.

Theorem 18. *If a and b are surreal numbers, then $\{A_L + b, a + B_L \mid A_R + b, a + B_R\}$ is well-formed.*

Proof. We will have proved the well-formedness of the sum, if we can prove all of the following:

$$A_L + b < A_R + b \tag{3.40}$$
$$A_L + b < a + B_R \tag{3.41}$$
$$a + B_L < A_R + b \tag{3.42}$$
$$a + B_L < a + B_R \tag{3.43}$$

(3.40) and (3.43) follow directly of the well-formedness of a and b combined with theorem 17.
(3.41) can be proved by noting that $A_L < a$, which means that

$$A_L + b < a + b, \tag{3.44}$$

and $b < B_R$, which means that

$$a + b < a + B_R. \tag{3.45}$$

By combining (3.44) and (3.45) using the transitive rule, we get (3.41).
(3.42) can be proved in a similar way. □

The second question we asked on page 27 was: Does surreal addition make sense? In other words, does it behave the way we expect addition to behave? We will address this question in two ways (using $\mathbb{R}$ to denote the set or real numbers and $\mathbb{S}$ to denote the set of surreal numbers):

- We will prove that $(\mathbb{S}, +)$ form a commutative (Abelian) group.
- We will prove that the Dali function is a homomorph mapping of $(\mathbb{R}, +_r)$ into $(\mathbb{S}, +_s)$, where $+_r$ denotes addition of real numbers and $+_s$ denotes addition of surreal numbers.

Let us address the first bullet point first. If we have a set, X, and an algebraic operator, $\star$, that operates on members of X, we say that $(X, \star)$ form a "group", if and only if

- the associative law holds: $(a \star b) \star c = a \star (b \star c)$, where a, b, and c are members of X,
- there exists a neutral element z in X, that is, an element with the property that $a \star z = a$ and $z \star a = a$ for all a in X, and
- every member, a, of X has an inverse member, $\bar{a}$, that is, a member with the property that $a \star \bar{a} = z$, where z is the neutral element.

If, furthermore, the commutative law holds ($a \star b = b \star a$), the group is said to be "commutative" or "Abelian".

Example. $(\mathbb{R}, +)$ is a commutative group with 0 as the neutral element and $-x$ as the inverse of x.

But $(\mathbb{R}, \times)$ is not a group. It does have a neutral element, 1, but not all members of $\mathbb{R}$ have an inverse, as there is no solution to the equation $0 \times x = 1$.

Theorem 19. *0 is the neutral element with respect to addition:* $0 + x = x$ *and* $x + 0 = x$.

Proof. Obviously, $0+0=0$. Furthermore we have:

$$\begin{aligned} 0+x &= \{\emptyset + x, 0 + X_L \mid \emptyset + x, 0 + X_R\} \\ &= \{0 + X_L \mid 0 + X_R\}. \end{aligned} \tag{3.46}$$

$\{0 + X_L \mid 0 + X_R\}$ is equal to x if theorem 19 is true for the parents of x. By induction we then have $0+x=x$.

In a similar way we can prove that $x+0=x$. □

Theorem 20. *The commutative law holds for surreal addition: $x+y=y+x$.*

Proof. It is obvious from the definition of addition that the commutative law is true for x and y if it is true for their parents. So all we have to prove is that $0+x=x+0$, which follows trivially from theorem 19. □

Theorem 21. *The associative law holds for surreal addition: $(x+y)+z=x+(y+z)$.*

Proof.

$$\begin{aligned} (x+y)+z &= \{(x+y)_L + z, (x+y) + Z_L \mid (x+y)_R + z, (x+y) + Z_R\} \\ &= \{(X_L + y) + z, (x + Y_L) + z, (x+y) + Z_L \mid (X_R + y) + z, (x + Y_R) + z, (x+y) + Z_R\}. \end{aligned} \tag{3.47}$$

$$\begin{aligned} x+(y+z) &= \{X_L + (y+z), x + (y+z)_L \mid X_R + (y+z), x + (y+z)_R\} \\ &= \{X_L + (y+z), x + (Y_L + z), x + (y + Z_L) \mid X_R + (y+z), x + (Y_R + z), x + (y + Z_R)\}. \end{aligned} \tag{3.48}$$

By comparing (3.47) and (3.48), we see that the associative law holds for x, y, and z, if it holds for their parents. And since it is obviously true if one of the numbers is replaced by zero, the theorem must be true. □

Theorem 22. *With respect to addition, every surreal number, x, has an inverse member, $-x$, such that $x+(-x)=0$. That number is $-x=\{-X_R \mid -X_L\}$.*

Proof. There are two things to prove here. First, we must prove that $-x$ exists, that is, we must prove that the proposed formula for $-x$ is a well-formed number. Secondly, we must prove that $x+(-x)=0$.

In order to understand the following arguments, it is necessary to notice that $(-X)_L = -X_R$ and $(-X)_R = -X_L$.

Proving that $-x$ is well-formed:

In order for $-x$ to be well-formed, we must prove that

$$-X_R < -X_L, \tag{3.49}$$

given that we know that $X_L < X_R$.

We will therefore prove that $a \leq b \;\Leftrightarrow\; -b \leq -a$. Once we have proved that, we can trivially deduce that $a < b \;\Leftrightarrow\; -b < -a$, and the well-formedness of $-x$ will follow immediately.

$a \leq b$ means that

$$\neg\exists\,\alpha \in A_L:\; b \leq \alpha \;\;\wedge\;\; \neg\exists\,\beta \in B_R:\; \beta \leq a. \tag{3.50}$$

Similarly $-b \leq -a$ means that

$$\neg\exists\,\xi \in (-B)_L:\; -a \leq \xi \;\;\wedge\;\; \neg\exists\,\eta \in (-A)_R:\; \eta \leq -b, \tag{3.51}$$

which is equivalent to

$$\neg\exists\,\xi \in -B_R:\; -a \leq \xi \;\;\wedge\;\; \neg\exists\,\eta \in -A_L:\; \eta \leq -b. \tag{3.52}$$

If we compare the first half of (3.50) with the second half of (3.52) and choosing $\alpha = -\eta$, and if we compare the second half of (3.50) with the first half of (3.52) and choosing $\xi = -\beta$, we see that $a \leq b \;\Leftrightarrow\; -b \leq -a$ is true if it is true for the parents of a and b.

In a similar manner we can prove that $a \leq 0 \;\Leftrightarrow\; 0 \leq -a$ if it is true for the parents of a.

Finally, we see that obviously $0 \leq -0$. Therefore, by induction, we have proved that $-x$ is well-formed.

Proving that $x + (-x) = 0$:

As always, we will prove the theorem by induction. For $x = 0$ we obviously have $-x = 0$, and $x + (-x) = 0$.

We will now assume that the theorem is true for x's parents and will then prove that it is then also true for x.

$$x + (-x) = \{X_L + (-x), x + (-X_R) \,|\, X_R + (-x), x + (-X_L)\} \tag{3.53}$$

Let us first consider the first element of the left set above:

$$X_L + (-x) = \{X_{LL} + (-x), X_L + (-X_R) \,|\, X_{LR} + (-x), X_L + (-X_L)\}, \tag{3.54}$$

where X_{LL} is the union of the left sets of every member of the left set of x, and X_{LR} is the union of the right sets of every member of the left set of x.

Since theorem 22 is assumed to be true for all parents of x, we have that $X_L + (-X_L) = 0$. We therefore have that the last element of the right set in (3.54) is 0, so it follows from theorem 5 that $X_L + (-x) < 0$.

Let us then consider the second element of the left set in (3.53):

$$x + (-X_R) = \{X_L + (-X_R), x + (-X_{RR}) \,|\, X_R + (-X_R), x + (-X_{RL})\}. \tag{3.55}$$

Again, by applying theorem 5 to (3.55) and the assumption that $X_R + (-X_R) = 0$, we conclude that $x + (-X_R) < 0$.

In a similar way we can prove that $X_R + (-x) > 0$ and $x + (-X_L) > 0$. So we can see from (3.53) that all members of the left set of $x + (-x)$ are less than 0, and all members of the right set of $x + (-x)$ are greater than 0. Therefore the value of $x + (-x) = 0$ since 0 is the oldest value between the two sets. $\square$

Notational convention. *Instead of $a + (-b)$, we will write $a - b$.*

We have now proved that $(\mathbb{S}, +)$ form a commutative group. In other words, it has the same algebraic properties as $(\mathbb{R}, +)$. Surely this is a minimum requirement if we want surreal addition to "make sense".

The second requirement we mentioned on page 30 was that the Dali function is a homomorph mapping of $(\mathbb{R}, +_r)$ into $(\mathbb{S}, +_s)$, where $+_r$ denotes addition of real numbers and $+_s$ denotes addition of surreal numbers. This simply means that we require that $\delta(a+b) = \delta(a) + \delta(b)$, where the first $+$ is addition of real numbers, and the second $+$ is addition of surreal numbers. Why is this a sensible requirement? Well, we use the Dali function to create the names 0, $\frac{1}{2}$, 1, 2, etc. for surreal numbers. We certainly want $2+3$ to be 5 both for real numbers and for their corresponding surreal counterparts. So, $\delta(2+3)$ gives us the surreal number whose name is 5, and $\delta(2) + \delta(3)$ is the sum of the two surreal numbers whose names are 2 and 3.

Theorem 23. *If a and b are real numbers, then $\delta(a + b) = \delta(a) + \delta(b)$.*

Proof. Case 1: a or b is zero:

We know from theorem 19 that $\{\,|\,\}$ is the neutral element for addition of surreal numbers, just as 0 is the neutral element for addition of real numbers. Therefore we have: $\delta(0 + b) = \delta(b) = \{\,|\,\} + \delta(b) = \delta(0) + \delta(b)$, and similarly for $\delta(a + 0)$.

Case 2: a and b are positive integers:

As usual we use induction, but this time not on the parents of surreal numbers but on ordinary integers.

According to (2.59), $\delta(a) = \{\delta(a-1)\,|\,\}$, $\delta(b) = \{\delta(b-1)\,|\,\}$, and $\delta(a+b) = \{\delta(a+b-1)\,|\,\}$.

We have already proved the theorem for $a = 0$ and $b = 0$. We will now assume that the theorem holds for $a - 1$ and $b - 1$. In other words, we assume that

$$\delta(a - 1 + b) = \delta(a - 1) + \delta(b) \tag{3.56}$$

$$\delta(a + b - 1) = \delta(a) + \delta(b - 1). \tag{3.57}$$

Based on this we have:

$$\begin{aligned}
\delta(a) + \delta(b) &= \{\delta(a)_L + \delta(b), \delta(a) + \delta(b)_L \,|\, \delta(a)_R + \delta(b), \delta(a) + \delta(b)_R\} \\
&= \{\delta(a-1) + \delta(b), \delta(a) + \delta(b-1)\,|\,\} \\
&= \{\delta(a-1+b), \delta(a+b-1)\,|\,\} \\
&= \delta(a+b).
\end{aligned} \tag{3.58}$$

Thus, by induction, the theorem is proved in this case.

Case 3: a and b are negative integers:

According to (2.59), $\delta(a) = \{\,|\,\delta(a+1)\}$, $\delta(b) = \{\,|\,\delta(b+1)\}$, and $\delta(a+b) = \{\,|\,\delta(a+b+1)\}$.

We have already proved the theorem for $a = 0$ and $b = 0$. We will now assume that the theorem holds for $a + 1$ and $b + 1$. In other words, we assume that

$$\delta(a + 1 + b) = \delta(a + 1) + \delta(b) \tag{3.59}$$

$$\delta(a + b + 1) = \delta(a) + \delta(b + 1). \tag{3.60}$$

Based on this we have:

$$\begin{aligned}\delta(a)+\delta(b) &= \{\delta(a)_L+\delta(b), \delta(a)+\delta(b)_L \mid \delta(a)_R+\delta(b), \delta(a)+\delta(b)_R\} \\ &= \{\,\mid \delta(a+1)+\delta(b), \delta(a)+\delta(b+1)\} \\ &= \{\,\mid \delta(a+1+b), \delta(a+b+1)\} \\ &= \delta(a+b).\end{aligned} \tag{3.61}$$

Thus, by induction, the theorem is proved in this case.

Case 4: a is a positive integer, b is a negative integer:
According to (2.59), $\delta(a)=\{\delta(a-1)\mid\}$, $\delta(b)=\{\,\mid\delta(b+1)\}$.
It is also important to note that according to (2.60)

$$\delta(a+b-1)<\delta(a+b)<\delta(a+b+1). \tag{3.62}$$

Depending on the sign of $a+b$ we have:

- If $a+b>0$, we have $\delta(a+b)=\{\delta(a+b-1)\mid\}$. If we combine this with (3.62) using theorem 11, we have $\delta(a+b)=\{\delta(a+b-1)\mid\delta(a+b+1)\}$.
- If $a+b=0$, we have $\delta(a+b)=\{\mid\}$. If we combine this with (3.62) using theorem 11, we have $\delta(a+b)=\{\delta(a+b-1)\mid\delta(a+b+1)\}$.
- If $a+b<0$, we have $\delta(a+b)=\{\,\mid\delta(a+b+1)\}$. If we combine this with (3.62) using theorem 11, we have $\delta(a+b)=\{\delta(a+b-1)\mid\delta(a+b+1)\}$.

So, in all cases we have

$$\delta(a+b)=\{\delta(a+b-1)\mid\delta(a+b+1)\}. \tag{3.63}$$

We have already proved theorem 23 for $a=0$ and $b=0$. We will now assume that the theorem holds for $a-1$ and $b+1$. In other words, we assume that

$$\delta(a-1+b)=\delta(a-1)+\delta(b) \tag{3.64}$$
$$\delta(a+b+1)=\delta(a)+\delta(b+1). \tag{3.65}$$

Based on this we have:

$$\begin{aligned}\delta(a)+\delta(b) &= \{\delta(a)_L+\delta(b), \delta(a)+\delta(b)_L \mid \delta(a)_R+\delta(b), \delta(a)+\delta(b)_R\} \\ &= \{\delta(a-1)+\delta(b) \mid \delta(a)+\delta(b+1)\} \\ &= \{\delta(a-1+b) \mid \delta(a+b+1)\} \\ &= \delta(a+b).\end{aligned} \tag{3.66}$$

Thus, by induction, the theorem is proved in this case.

Case 5: a is a positive integer, b is an irreducible fraction of the form $\frac{j}{2^k}$, where j and k are integers, and $k>0$:

According to (2.59), $\delta(a) = \{\delta(a-1)\,|\,\}$ and $\delta(b) = \{\delta(\frac{j-1}{2^k})\,|\,\delta(\frac{j+1}{2^k})\}$. We also have

$$\begin{aligned} a+b &= a+\frac{j}{2^k} \\ &= \frac{2^k a+j}{2^k}, \end{aligned} \tag{3.67}$$

and therefore

$$\delta(a+b) = \Big\{\delta\Big(\frac{2^k a+j-1}{2^k}\Big) \;\Big|\; \delta\Big(\frac{2^k a+j+1}{2^k}\Big)\Big\}. \tag{3.68}$$

We have already proved theorem 23 for the case where $a = 0$, and for the case where b is an integer, that is, the case where $k = 0$. Now we assume that the theorem is true for $a-1$ and $k-1$, and we will prove that it is then also true for a and k.

We have:

$$\begin{aligned} \delta(a)+\delta(b) &= \{\delta(a)_L+\delta(b), \delta(a)+\delta(b)_L \,|\, \delta(a)_R+\delta(b), \delta(a)+\delta(b)_R\} \\ &= \Big\{\delta(a-1)+\delta\Big(\frac{j}{2^k}\Big), \delta(a)+\delta\Big(\frac{j-1}{2^k}\Big) \;\Big|\; \delta(a)+\delta\Big(\frac{j+1}{2^k}\Big)\Big\} \\ &= \Big\{\delta\Big(a-1+\frac{j}{2^k}\Big), \delta\Big(a+\frac{j-1}{2^k}\Big) \;\Big|\; \delta\Big(a+\frac{j+1}{2^k}\Big)\Big\}. \end{aligned} \tag{3.69}$$

The last operation was possible because

- $\frac{j-1}{2^k}$ and $\frac{j+1}{2^k}$ can be reduced by at least 2, which changes the denominator to 2^{k-1} or less, which means that we can apply our assumption about the validity of the theorem; and because
- we assume the theorem to be true for $a-1$.

Further manipulation leads us to:

$$\begin{aligned} \delta(a)+\delta(b) &= \Big\{\delta\Big(\frac{2^k a-2^k+j}{2^k}\Big), \delta\Big(\frac{2^k a+j-1}{2^k}\Big) \;\Big|\; \delta\Big(\frac{2^k a+j+1}{2^k}\Big)\Big\} \\ &= \Big\{\delta\Big(\frac{2^k a+j-1}{2^k}\Big) \;\Big|\; \delta\Big(\frac{2^k a+j+1}{2^k}\Big)\Big\}, \end{aligned} \tag{3.70}$$

where the last simplification was possible because $k \geq 0$ and hence $2^k \geq 1$.

By comparing this final result with (3.68) we've proved that $\delta(a)+\delta(b) = \delta(a+b)$ in this case.

Case 6: a is a negative integer, b is an irreducible fraction of the form $\frac{j}{2^k}$, where j and k are integers, and $k > 0$:

The proof of this case is quite similar to the proof of case 5, and will not be included here.

Case 7: a is an irreducible fraction of the form $\frac{j_a}{2^{k_a}}$, where j_a and k_a are integers, and $k_a > 0$, b is an irreducible fraction of the form $\frac{j_b}{2^{k_b}}$, where j_b and k_b are integers, and $k_b > 0$:

According to (2.59), $\delta(a) = \{\delta(\frac{j_a-1}{2^{k_a}})\,|\,\delta(\frac{j_a+1}{2^{k_a}})\}$ and $\delta(b) = \{\delta(\frac{j_b-1}{2^{k_b}})\,|\,\delta(\frac{j_b+1}{2^{k_b}})\}$. Without loss of generality, we can assume that $k_a \geq k_b$, which means that

$$\begin{aligned} a+b &= \frac{j_a}{2^{k_a}}+\frac{j_b}{2^{k_b}} \\ &= \frac{j_a+2^{k_a-k_b}j_b}{2^{k_a}}, \end{aligned} \tag{3.71}$$

and therefore

$$\delta(a+b) = \Big\{\delta\Big(\frac{j_a + 2^{k_a-k_b} j_b - 1}{2^{k_a}}\Big) \;\Big|\; \delta\Big(\frac{j_a + 2^{k_a-k_b} j_b + 1}{2^{k_a}}\Big)\Big\}. \tag{3.72}$$

We have already proved theorem 23 for the case where a and b are integers, that is, the case where $k_a = 0$ and $k_b = 0$. Now we assume that the theorem is true for $k_a - 1$ and $k_b - 1$, and we will prove that it is then also true for k_a and k_b.

We have:

$$\begin{aligned}
\delta(a) + \delta(b) &= \{\delta(a)_L + \delta(b), \delta(a) + \delta(b)_L \,|\, \delta(a)_R + \delta(b), \delta(a) + \delta(b)_R\} \\
&= \Big\{\delta\Big(\frac{j_a - 1}{2^{k_a}}\Big) + \delta\Big(\frac{j_b}{2^{k_b}}\Big), \delta\Big(\frac{j_a}{2^{k_a}}\Big) + \delta\Big(\frac{j_b - 1}{2^{k_b}}\Big) \;\Big| \\
&\qquad\qquad \delta\Big(\frac{j_a + 1}{2^{k_a}}\Big) + \delta\Big(\frac{j_b}{2^{k_b}}\Big), \delta\Big(\frac{j_a}{2^{k_a}}\Big) + \delta\Big(\frac{j_b + 1}{2^{k_b}}\Big)\Big\} \\
&= \Big\{\delta\Big(\frac{j_a - 1}{2^{k_a}} + \frac{j_b}{2^{k_b}}\Big), \delta\Big(\frac{j_a}{2^{k_a}} + \frac{j_b - 1}{2^{k_b}}\Big) \;\Big|\; \delta\Big(\frac{j_a + 1}{2^{k_a}} + \frac{j_b}{2^{k_b}}\Big), \delta\Big(\frac{j_a}{2^{k_a}} + \frac{j_b + 1}{2^{k_b}}\Big)\Big\}.
\end{aligned} \tag{3.73}$$

The last operation was possible because $\frac{j_a - 1}{2^{k_a}}$ and similar fractions can be reduced by at least 2, which changes the denominator to $2^{k_a - 1}$ or less, which means that we can apply our assumption about the validity of the theorem.

Further manipulation leads us to:

$$\begin{aligned}
\delta(a) + \delta(b) &= \Big\{\delta\Big(\frac{(j_a - 1) + 2^{k_a-k_b} j_b}{2^{k_a}}\Big), \delta\Big(\frac{j_a + (j_b - 1)2^{k_a-k_b}}{2^{k_a}}\Big) \;\Big| \\
&\qquad\qquad \delta\Big(\frac{(j_a + 1) + 2^{k_a-k_b} j_b}{2^{k_a}}\Big), \delta\Big(\frac{j_a + (j_b + 1)2^{k_a-k_b}}{2^{k_a}}\Big)\Big\} \\
&= \Big\{\delta\Big(\frac{j_a + 2^{k_a-k_b} j_b - 1}{2^{k_a}}\Big), \delta\Big(\frac{j_a + 2^{k_a-k_b} j_b - 2^{k_a-k_b}}{2^{k_a}}\Big) \;\Big| \\
&\qquad\qquad \delta\Big(\frac{j_a + 2^{k_a-k_b} j_b + 1}{2^{k_a}}\Big), \delta\Big(\frac{j_a + 2^{k_a-k_b} j_b + 2^{k_a-k_b}}{2^{k_a}}\Big)\Big\} \\
&= \Big\{\delta\Big(\frac{j_a + 2^{k_a-k_b} j_b - 1}{2^{k_a}}\Big) \;\Big|\; \delta\Big(\frac{j_a + 2^{k_a-k_b} j_b + 1}{2^{k_a}}\Big)\Big\},
\end{aligned} \tag{3.74}$$

where the last simplification was possible because $k_a \geq k_b$ and hence $2^{k_a - k_b} \geq 1$.

By comparing this final result with (3.72) we've proved that $\delta(a) + \delta(b) = \delta(a + b)$ in this – final – case. □

Chapter 4

Multiplication

In this chapter, we use the usual convention of writing the product of a and b either as ab or as $a \times b$.

Definition 4. *The product of two surreal numbers, a and b, is defined thus:*

$$ab = \{A_L b + aB_L - A_L B_L, A_R b + aB_R - A_R B_R \mid A_L b + aB_R - A_L B_R, A_R b + aB_L - A_R B_L\}. \tag{4.1}$$

As with addition, we ask ourselves two questions:

- Is the definition legal? That is, does it produce well-formed surreal numbers?
- Does the definition make sense?

The proof of the first bullet point will be omitted here. We will address the second bullet point in much the same way we did with addition:

- We will prove that $(\mathbb{S} \setminus \{0\}, \times)$ form a commutative (Abelian) group, and
- We will prove that the Dali function is a homomorph mapping of $(\mathbb{R}, \times_r)$ into $(\mathbb{S}, \times_s)$, where $\times_r$ denotes multiplication of real numbers and $\times_s$ denotes multiplication of surreal numbers.

0 plays a special role in multiplication:

Theorem 24. $0 \times x = x \times 0 = 0$.

Proof.

$$\begin{aligned} 0x &= \{\emptyset x + 0X_L - \emptyset X_L, \emptyset x + 0X_R - \emptyset X_R \mid \emptyset x + 0X_R - \emptyset X_R, \emptyset x + 0X_L - \emptyset X_L\} \\ &= \{\,\mid\,\} \\ &= 0. \end{aligned} \tag{4.2}$$

$x0 = 0$ can be proved in a similar manner. □

Theorem 25. *1 is the neutral element with respect to multiplication: $1 \times x = x$ and $x \times 1 = x$.*

Proof.

$$\begin{aligned}1x &= \{0x + 1X_L - 0X_L, \emptyset x + 1X_R - \emptyset X_R \mid 0b + 1X_R - 00_R, \emptyset x + 1X_L - \emptyset X_L\} \\ &= \{1X_L \mid 1X_R\}.\end{aligned} \tag{4.3}$$

$\{1X_L \mid 1X_R\}$ is equal to x if theorem 25 is true for the parents of x. Thus induction shows us that the theorem is true if it is true for $x = 0$, which it obviously is. So $1x = x$.

In a similar way we can prove that $x1 = x$. □

Theorem 26. *The commutative law holds for surreal multiplication: $xy = yx$.*

Proof. It is obvious from the definition of multiplication that the commutative law is true for x and y if it is true for their parents. So all we have to prove is that $0x = x0$, which follows trivially from theorem 24. □

Theorem 27. *The associative law holds for surreal multiplication: $(xy)z = x(yz)$.*

Proof. Omitted. □

Theorem 28. *The distributive law holds for surreal multiplication and addition: $x(y+z) = xy+xz$.*

Proof.

$$\begin{aligned}x(y+z) = \{&X_L(y+z) + x(y+z)_L - X_L(y+z)_L, X_R(y+z) + x(y+z)_R - X_R(y+z)_R \mid \\ &X_L(y+z) + x(y+z)_R - X_L(y+z)_R, X_R(y+z) + x(y+z)_L - X_R(y+z)_L\} \\ = \{&X_L(y+z) + x\{Y_L + z, y + Z_L\} - X_L\{Y_L + z, y + Z_L\}, \\ &X_R(y+z) + x\{Y_R + z, y + Z_R\} - X_R\{Y_R + z, y + Z_R\} \mid \\ &X_L(y+z) + x\{Y_R + z, y + Z_R\} - X_L\{Y_R + z, y + Z_R\}, \\ &X_R(y+z) + x\{Y_L + z, y + Z_L\} - X_R\{Y_L + z, y + Z_L\}\} \\ = \{&X_L(y+z) + x(Y_L + z) - X_L(Y_L + z), X_L(y+z) + x(y + Z_L) - X_L(y + Z_L), \\ &X_R(y+z) + x(Y_R + z) - X_R(Y_R + z), X_R(y+z) + x(y + Z_R) - X_R(y + Z_R) \mid \\ &X_L(y+z) + x(Y_R + z) - X_L(Y_R + z), X_L(y+z) + x(y + Z_R) - X_L(y + Z_R), \\ &X_R(y+z) + x(Y_L + z) - X_R(Y_L + z), X_R(y+z) + x(y + Z_L) - X_R(y + Z_L)\}.\end{aligned} \tag{4.4}$$

$$\begin{aligned}xy + xz &= \{(xy)_L + xz, xy + (xz)_L \mid (xy)_R + xz, xy + (xz)_R\} \\ &= \{\{X_L y + xY_L - X_L Y_L, X_R y + xY_R - X_R Y_R\} + xz, \\ &\quad xy + \{X_L z + xZ_L - X_L Z_L, X_R z + xZ_R - X_R Z_R\} \mid \\ &\quad \{X_L y + xY_R - X_L Y_R, X_R y + xY_L - X_R Y_L\} + xz, \\ &\quad xy + \{X_L z + xZ_R - X_L Z_R, X_R z + xZ_L - X_R Z_L\}\} \\ &= \{X_L y + xY_L - X_L Y_L + xz, X_R y + xY_R - X_R Y_R + xz, \\ &\quad xy + X_L z + xZ_L - X_L Z_L, xy + X_R z + xZ_R - X_R Z_R \mid \\ &\quad X_L y + xY_R - X_L Y_R + xz, X_R y + xY_L - X_R Y_L + xz, \\ &\quad xy + X_L z + xZ_R - X_L Z_R, xy + X_R z + xZ_L - X_R Z_L\}.\end{aligned} \tag{4.5}$$

By comparing (4.4) and (4.5), we see that the distributive law holds for x, y, and z, if it holds for their parents. And since it is obviously true if one of the numbers is replaced by zero, the theorem must be true. $\square$

Theorem 29. *With respect to multiplication, every surreal number, x, except 0, has an inverse member, $\frac{1}{x}$, such that $x \times \frac{1}{x} = 1$.*

The proof of this theorem is based on principles we will discuss in the following chapter. Up until now we have only met fractions where the denominator is a power of two. We haven't yet learned the surreal number whose value is, for example, $\frac{1}{3}$.

We have now established that $(\mathbb{S} \setminus \{0\}, \times)$ form a commutative group, and that the distributive law holds. In total, we have shown that surreal numbers with addition and multiplication form what mathematicians call a "ring". So multiplication of the surreal numbers has the same algebraic properties as multiplication of the real numbers. Surely this is a minimum requirement if we want surreal multiplication to "make sense".

The second requirement we mentioned on page 37 was that the Dali function is a homomorph mapping of $(\mathbb{R}, \times_r)$ into $(\mathbb{S}, \times_s)$, where $\times_r$ denotes multiplication of real numbers and $\times_s$ denotes multiplication of surreal numbers.

Theorem 30. *If a and b are real numbers, then $\delta(ab) = \delta(a)\delta(b)$.*

Proof. Omitted. $\square$

Chapter 5

"To Infinity and Beyond"

So far, surreal numbers haven't really given us anything new, have they? They've only given us integers and fractions whose denominators are powers of two. Although it certainly is fascinating that so much can be built on such a simple basis, we haven't got anything new out of playing with surreal numbers.

This chapter will change all that.

We can define the set of integers, $\mathbb{Z}$, thus:

$$\begin{aligned} &0 \in \mathbb{Z} \\ &n \in \mathbb{Z} \;\Rightarrow\; \{n\,|\,\} \in \mathbb{Z} \\ &n \in \mathbb{Z} \;\Rightarrow\; \{\,|\,n\} \in \mathbb{Z}. \end{aligned} \tag{5.1}$$

According to (5.1), 0 is in $\mathbb{Z}$, and therefore $1 = \{0\,|\,\}$ and $-1 = \{\,|\,0\}$ are both in $\mathbb{Z}$, and therefore $2 = \{1\,|\,\}$ and $-2 = \{\,|\,-1\}$ are both in $\mathbb{Z}$, etc.

Note that we've now completely dropped the distinction between ordinary integers and surreal numbers that have integer names (that is, surreal numbers that are the result of applying the Dali function to ordinary integers).

$\mathbb{Z}$, as defined above, is clearly a set of surreal numbers, so we can create a new surreal number thus:

$$\{\mathbb{Z}\,|\,\}.$$

It is obvious that this is a valid surreal number, as its left and right sets are sets of surreal numbers, and as it is obviuosly well-formed.

But what is its value? According to theorem 5, $\{\mathbb{Z}\,|\,\}$ is a number that is greater than all integers. Therefore, its value is infinity!

I would dearly have loved to call this number ∞, because that would later have allowed me to write such wonderful expressions as $\infty + 5$, $\frac{\infty}{2}$, and $\sqrt{\infty}$. But, alas, mathematical convention prevents me from doing so. Infinity is a very vague concept, and mathematicians distinguish between many different types of infinities. It is customary to use the Greek letter ω to denote the number $\{\mathbb{Z}\,|\,\}$. ω is an "ordinal". It is one of a special type of infinities and accurately matches our $\{\mathbb{Z}\,|\,\}$.

We will not here dig into the theory of ordinals. Suffice it to say that all ordinals can be expressed as surreal numbers.

So, with $\omega \equiv \{\mathbb{Z}\,|\,\}$ in place, we can proceed to calculate a few other numbers, using the definition of addition, subtraction, and multiplication:

$$\begin{aligned} \omega - 1 &= \{\mathbb{Z} - 1\,|\,\omega - 0\} \\ &= \{\mathbb{Z}\,|\,\omega\}. \end{aligned} \tag{5.2}$$

Here we've used the fact that if you subtract 1 from the set of all integers, you still get the set of all integers.

We also have:

$$\begin{aligned} \omega + 1 &= \{\mathbb{Z} + 1, \omega + 0\,|\,\} \\ &= \{\omega\,|\,\}. \end{aligned} \tag{5.3}$$

Since ω is greater than all integers, we can drop $\mathbb{Z}$ from the expression. So now you know what infinity minus one and infinity plus one are.

Similarly we have:

$$\omega + 2 = \{\omega + 1\,|\,\} \tag{5.4}$$
$$\omega + 3 = \{\omega + 2\,|\,\} \tag{5.5}$$
$$\omega - 2 = \{\mathbb{Z}\,|\,\omega - 1\} \tag{5.6}$$
$$\omega - 3 = \{\mathbb{Z}\,|\,\omega - 2\} \tag{5.7}$$
$$\omega + \omega = \{\omega + \mathbb{Z}\,|\,\}. \tag{5.8}$$

What is $\omega + \mathbb{Z}$? We've already learned the values of $\omega + 1$, $\omega + 2$, $\omega + 3$, $\omega - 1$, $\omega - 2$, $\omega - 3$, etc. So, $\omega + \mathbb{Z}$ is the set of all these numbers.

We also have:

$$2\omega = \{\omega + \mathbb{Z}\,|\,\} \tag{5.9}$$
$$3\omega = \{2\omega + \mathbb{Z}\,|\,\} \tag{5.10}$$
$$4\omega = \{3\omega + \mathbb{Z}\,|\,\} \tag{5.11}$$
$$\omega^2 = \{\omega, 2\omega, 3\omega, 4\omega, \ldots\,|\,\} \tag{5.12}$$
$$\omega^\omega = \{\omega, \omega^2, \omega^3, \omega^4, \ldots\,|\,\} \tag{5.13}$$
$$\frac{\omega}{2} = \{\mathbb{Z}\,|\,\omega - \mathbb{Z}\} \tag{5.14}$$
$$\sqrt{\omega} = \{\mathbb{Z}\,|\,\omega, \frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \ldots\} \tag{5.15}$$
$$-\omega = \{\,|\,\mathbb{Z}\}. \tag{5.16}$$

Of course, all of these numbers have many other representatives. Instead of $\{\mathbb{Z}\,|\,\}$, we could also write ω as $\{1, 2, 3, \ldots\,|\,\}$ or $\{2, 4, 6, 8, \ldots\,|\,\}$ or $\{P\,|\,\}$, where P is the set of all prime numbers whose last three digits are 593. Theorem 9 tells us that we can remove any member from the left set of ω as long as we leave a member that is larger than the one we remove.

So we see that the surreal numbers contain a lot of numbers that are not ordinary real numbers.

But there's more... Consider this surreal number:

$$\varepsilon = \{0 \,|\, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}. \tag{5.17}$$

This, again, is a well-formed number, but its value is greater than 0 and less than all positive fractions. This is an *infinitesimal* number – not zero, but smaller than the smallest fraction.

It turns out that $\varepsilon = \frac{1}{\omega}$, so here we have the reciprocal of infinity. Of course, we can find many other interesting numbers:

$$\varepsilon + 1 = \{1 \,|\, 2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \ldots\} \tag{5.18}$$

is a number greater than 1 but smaller than any real number above 1.

And we have

$$2\varepsilon = \{\varepsilon \,|\, 1+\varepsilon, \frac{1}{2}+\varepsilon, \frac{1}{4}+\varepsilon, \frac{1}{8}+\varepsilon, \frac{1}{16}+\varepsilon, \ldots\} \tag{5.19}$$

$$\frac{\varepsilon}{2} = \{0 \,|\, \varepsilon\} \tag{5.20}$$

$$\sqrt{\varepsilon} = \{\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, \ldots \,|\, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}. \tag{5.21}$$

It is hardly surprising that it has been difficult to find a useful definition of differentiation and integration using surreal numbers. What exactly does $\frac{dy}{dx}$ mean when x and y themselves can be a fraction of an infinitesimal?

But there's more... So far, all the surreal fractions we've created have been of the form $\frac{j}{2^k}$, where j and k are integers, and $k \geq 0$. Such fractions are called *dyadic*.

Now consider the surreal number $t = \{T_L \,|\, T_R\}$, where

- T_L is the set of all dyadic fractions, $\frac{j}{2^k}$, where $3j < 2^k$.
- T_R is the set of all dyadic fractions, $\frac{j}{2^k}$, where $3j > 2^k$.

What is the value of t? We see that $\delta(x)$ is a member of the left set if $\delta(x)$ is a dyadic fraction and $x < \frac{1}{3}$. Similarly, $\delta(x)$ is a member of the right set if $\delta(x)$ is a dyadic fraction and $x > \frac{1}{3}$. It seems reasonable to say that we have finally found the surreal number whose value is $\frac{1}{3}$. Or, if you like,

$$\delta(\tfrac{1}{3}) = \{T_L \,|\, T_R\}, \tag{5.22}$$

where T_L and T_R are defined above.

But perhaps t is smaller than $\frac{1}{3}$ by an infinitesimal number? In that case it would still lie between T_L and T_R. Can we prove that we've actually found $\frac{1}{3}$? Fortunately, we can.

Proof. We first notice that t is not itself a dyadic fraction. If it were so, it would have been a member of either T_L or T_R. We know then that t is greater than all dyadic fractions that are less than $\frac{1}{3}$ and t is less than all dyadic fractions that are greater than $\frac{1}{3}$.

Let us now calculate $2t$:

$$2t = t + t = \{T_L + t \,|\, T_R + t\}. \tag{5.23}$$

Since T_L consists of numbers that are less than $\frac{1}{3}$ and T_R consists of numbers that are greater than $\frac{1}{3}$, $2t$ must be greater than numbers that are less than $\frac{2}{3}$ and less than numbers that are greater than $\frac{2}{3}$.

Now let us calculate $3t$:

$$3t = 2t + t = \{T_L + 2t, 2T_L + t \,|\, T_R + 2t, 2T_R + t\}. \tag{5.24}$$

We see that both members of the left set are less than 1, and both members of the right set are greater than 1. We know from corollary 13 that $3t$ must have the value of the oldest surreal number between the values in the left set and the values in the right set; $3t$ must have the value 1. Therefore $t = \frac{1}{3}$. □

On page 24 we asked for the value of $\delta(\frac{1}{3})$. Now, we've found it! And isn't it marvelous that in the realm of surreal numbers, $\frac{1}{3}$ is just as weird as ω or ε? You need infinite sets to construct all these numbers.

But now we can also easily find the surreal number whose value is π (or rather, $\delta(\pi)$):

$$\begin{aligned} \pi = \{L \,|\, R\},\ &\text{where } L \text{ contains all dyadic fractions that are less than } \pi, \\ &\text{and } R \text{ contains all dyadic fractions that are greater than } \pi. \end{aligned} \tag{5.25}$$

Of course, the constituent sets in these surreal numbers carry a lot of excess baggage; they contain a lot of uninteresting members. It is therefore frequently desirable to find some useful subset of the two sets. Again, that is fairly easy.

Let us again consider $\frac{1}{3}$. Now take a dyadic fraction $\frac{j}{2^k}$. For every value of k, there is some value of j for which this fraction is closest to $\frac{1}{3}$. For example, $\frac{3}{8}$ is closer to $\frac{1}{3}$ than any other fraction $\frac{j}{8}$. We can now weed out in the fractions that make up T_L and T_R, and only include these "best-fit" fractions:

$$\frac{1}{3} = \left\{\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \ldots \;\middle|\; \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \frac{43}{128}, \ldots\right\}. \tag{5.26}$$

And similarly:

$$\pi = \left\{\frac{3}{1}, \frac{25}{8}, \frac{201}{64}, \ldots \;\middle|\; \frac{13}{4}, \frac{101}{32}, \frac{3217}{1024}, \ldots\right\}. \tag{5.27}$$

Note that these values for $\frac{1}{3}$ and π are not "better" or more correct than (5.22) and (5.25). They may just prove more "useful" in some contexts.

In the previous chapters we have seen formulas for addition, subtraction, and multiplication of surreal numbers, but no formula for division. Below you will find a way to calculate $\frac{1}{x}$ for positive values of x, which – together with the formula for multiplication – is enough to calculate $\frac{y}{x}$.

$\frac{1}{x} = \{L \mid R\}$, where L and R are defined thus:

$$\begin{aligned}
&0 \in L \\
&\lambda \in L \;\Rightarrow\; \frac{1 + (X_R - x)\lambda}{X_R} \in L \\
&\lambda \in L \;\Rightarrow\; \frac{1 + (X_L - x)\lambda}{X_L} \in R \\
&\rho \in R \;\Rightarrow\; \frac{1 + (X_L - x)\rho}{X_L} \in L \\
&\rho \in R \;\Rightarrow\; \frac{1 + (X_R - x)\rho}{X_R} \in R.
\end{aligned} \tag{5.28}$$

Only positive members of X_L and X_R are used.

The proof is omitted here.

Example. Let's express $\frac{1}{5}$ as a surreal number. We have $x = 5$, and therefore $X_L = \{4\}$ and $X_R = \emptyset$. (5.28) therefore reduces to:

$$\begin{aligned}
&0 \in L \\
&\lambda \in L \;\Rightarrow\; \frac{1 - \lambda}{4} \in R \\
&\rho \in R \;\Rightarrow\; \frac{1 - \rho}{4} \in L
\end{aligned} \tag{5.29}$$

So we get successively: $0 \in L$, $\frac{1}{4} \in R$, $\frac{3}{16} \in L$, $\frac{13}{64} \in R$, etc.

Note that (5.28) doesn't necessarily produce what we might consider the "simplest" representative for a surreal number. If you use (5.28) to calculate $\frac{1}{4}$, you'll not get $\{0 \mid \frac{1}{2}\}$.

Calculating y^x where x is an integer is very simple when we know the formula for multiplication and division. Calculating y^x when x is not an integer is more complex.

A formula for $\exp(x)$ (or e^x) exists, and it is given here without proof:

$$\exp(x) = \{0, T_1, T_2 \mid T_3, T_4\}, \tag{5.30}$$

where

$$T_1 = \exp(X_L) f(x - X_L, n), \quad \text{where } n \text{ assumes all positive integer values,} \tag{5.31}$$

$$T_2 = \exp(X_R) f(x - X_R, n), \quad \text{where } n \text{ assumes all positive odd integer values,} \tag{5.32}$$

$$T_3 = \frac{\exp(X_R)}{f(X_R - x, n)}, \quad \text{where } n \text{ assumes all positive integer values,} \tag{5.33}$$

$$T_4 = \frac{\exp(X_L)}{f(X_L - x, n)}, \quad \text{where } n \text{ assumes all positive odd integer values.} \tag{5.34}$$

The function $f(y, n)$ is calculated thus:

$$f(y, n) = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!}. \tag{5.35}$$

For $y < 0$, only positive values of $f(y, n)$ are used in the formulas above.

If you are interested in a detailed treatment of this complex formula, please see Harry Gonshor's book which is listed in chapter 7.

Chapter 6

Pseudo-numbers and Games

We will now investigate what happens if we drop the requirement made in definition 1 on page 6 that surreal numbers must be well-formed.

Knuth [2] called numbers that are not well-formed "pseudo-numbers".

Investigation of pseudo-numbers leads to some interesting results. Obviously, any theorem that we have proved using the well-formedness of surreal numbers may not be true for pseudo-numbers.

Let us consider the pseudo-number $\{5\,|\,3\}$. If we compare this number to a suitable collection of surreal numbers, we find that

$$x < 3 \;\Rightarrow\; x < \{5\,|\,3\}, \tag{6.1}$$

$$x > 5 \;\Rightarrow\; \{5\,|\,3\} < x, \tag{6.2}$$

$$3 \le x \le 5 \;\Rightarrow\; x \not\le \{5\,|\,3\} \;\wedge\; \{5\,|\,3\} \not\le x. \tag{6.3}$$

So if x lies between 3 and 5, we find that $\{5\,|\,3\}$ is neither less than nor greater than nor equal to x.

As numbers go, pseudo-numbers are obviously not very useful, so it makes sense to require that surreal numbers be well-formed. However, pseudo-numbers play a useful role in game theory.

We will not here dig much into the application of surreal numbers and pseudo-numbers to game theory; but a few things are worth noting.

We will consider games played by two players called *Left* and *Right*. The games involve no luck and no hidden information. Chess is an example of such a game, backgammon and Stratego are not[1].

Let x be a position in a game. If *Left* is to move, he can turn the position x into a number of other positions, x_1, x_2, x_3. If *Right* is to move, she can turn the position x into a number of other positions, x'_1, x'_2, x'_3. We will write this thus:

$$x = \{x_1, x_2, x_3\,|\,x'_1, x'_2, x'_3\}. \tag{6.4}$$

Here, x is written as a surreal number or a pseudo-number whose left set consists of the positions that can be reached if *Left* is to move, and whose right set consists of the positions that can be reached if *Right* is to move.

[1]There are other problems with chess, but we'll ignore them here.

If the next player to move finds that he has lost, he has no moves to make. So, for example, a number whose right set is empty denotes a position where *Right* has lost if she is the player that should make the next move.

We can now make a few observations:

- $0 = \{\,|\,\}$ is a position where the next player to move has lost.
- $1 = \{0\,|\,\}$ is a position which *Left* will win – either because *Right* is about to move but has no legal move, or because *Left* is about to move and creates the position 0, in which *Right* has lost.
- $-1 = \{\,|\,0\}$ is a position in which *Right* will win.
- $\{0\,|\,0\}$ is a position where the next player to move will win, because the move will lead to a position where the next player to move has lost.

Conway in [1] calls $\{0\,|\,0\}$ "star" and denotes it by the symbol $*$. The pseudo-number $\{*\,|\,*\}$ is a position where the next player to move will lose, because the move will lead to position $\{0\,|\,0\}$, in which the next player to move will win. Thus, both $\{*\,|\,*\}$ and $\{\,|\,\}$ identify a position in which the next player to move will lose.

The amazing thing now is that if we compare $\{*\,|\,*\}$ with $\{\,|\,\}$ using our well-known definitions of $\leq$ and $=$, we will find that $\{*\,|\,*\} = \{\,|\,\}$.

Chapter 7

References

[1] J. H. Conway: *On numbers and games*, second edition, A. K. Peters, 2001. ISBN 1-56881-127-6.
This is *the* book on surreal numbers. If you read nothing else about surreal numbers, this is the book you should read. The book contains an extensive description of surreal numbers, pseudo-numbers, and their application to game theory.

[2] D. E. Knuth: *Surreal Numbers*. Addison-Wesley, 1974.
Knuth's book tells about how a young man and woman on a romantic trip to a secluded place at the Indian Ocean discover an ancient description of numbers. They proceed to prove various theorems about these numbers.
Instead of $\{L \mid R\}$, Knuth writes a surreal number thus: (L, R). It should also be noted that he uses $=$ where Conway and I use $\equiv$, and vice versa.

[3] Harry Gonshor: *An Introduction to the Theory of Surreal Numbers*. London Mathematical Society, Lecture Note Series 110. Cambridge University Press, 1986. ISBN 0-521-31205-1.
Gonshor's book is the mathematically most demanding of the books mentioned here. He expands the theory of surreal number beyond what the other two books do, basing his work in part on a different notation than the one we've used here. Gonshor uses "$+$" to denote 1, "$++$" to denote 2, "$++-$" to denote $1\frac{1}{2}$, "$--$" to denote -2, etc. This makes his proofs easier, but the basis less elegant, in my opinion.
Instead of $\{L \mid R\}$, Gonshor uses the notation $L|R$.

Index